On (twisted) Lawrence-Krammer representations.

Anatole Castella

ON (TWISTED) LAWRENCE-KRAMMER REPRESENTATIONS

ANATOLE CASTELLA

Abstract. LK-representations (LK for Lawrence-Krammer) are linear representations of Artin-Tits monoids and groups of small type, which are of particular interest since they are known to be faithful for the monoids, and for the groups when the type is spherical, under some (strong) conditions on the defining ring $\mathfrak{R}$.

For a fixed small type, they are parametrized by an $\mathfrak{R}$-module $\mathscr{F}$ for the monoid, and by a subset $\mathscr{F}_{\mathrm{gr}}$ of $\mathscr{F} \setminus \{0\}$ for the group. It is known that $\mathscr{F}_{\mathrm{gr}}$ is non-empty if the type has no triangle, *i.e.* no subgraph of affine type $\tilde{A}_2$, and more precisely that $\mathscr{F} = \mathfrak{R}$ and $\mathscr{F}_{\mathrm{gr}} = \mathfrak{R}^\times$ if the type is spherical. Moreover, faithful "twisted" LK-representations for the non-small (spherical) crystallographic types have been constructed in [Digne, *On the linearity of Artin Braid groups.* J. Algebra **268**, (2003) 39-57].

The first aim of this paper is to explicit $\mathscr{F}$ and $\mathscr{F}_{\mathrm{gr}}$ for any affine and small type : we establish that $\mathscr{F} = \mathfrak{R}^{\mathbb{N}}$ and $\mathscr{F}_{\mathrm{gr}} = \mathfrak{R}^\times \times \mathfrak{R}^{\mathbb{N}_{\geqslant 1}}$, and since this holds in particular for $\tilde{A}_2$, this shows that $\mathscr{F}_{\mathrm{gr}}$ can be non-empty for a graph with triangles. The second aim is to generalize the construction of *op.cit.* in order to provide faithful twisted LK-representations for any Artin-Tits monoid that appears as the submonoid of fixed elements of an Artin-Tits monoid of small type under a group of graph automorphisms ; in particular, we thus get three twisted LK-representations for the Coxeter type B_n (among which the one constructed in *op.cit.*) and we finally show that they are pairwise non-equivalent, at least for the main choice of $\mathfrak{R}$.

Introduction

In the early 2000's, Krammer defined by explicit formulas a linear representation of the braid group on a free $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$-module of dimension the number of positive roots in the associated root system, and proved its faithfulness [16, 17] (see also [1]). This construction and the proof of faithfulness have been generalized by Cohen and Wales, and independently by Digne, to the Artin-Tits groups of spherical and small type [8, 12], and then to all the Artin-Tits monoids of small type by Paris [21] (see also [14] for a short proof of the faithfulness). Since an homological version of this representation first appears in the work of Lawrence [18], those representations are commonly called *Lawrence-Krammer* (LK for short).

In fact, LK-representations can be defined over an arbitrary unitary commutative ring $\mathfrak{R}$, as far as we do not require their faithfulness (see subsection 2.2 below for a faithfulness criterion on $\mathfrak{R}$). This slight generalization will give us some more insight on what is really needed in the construction, and will at least simplify some computations. Let us be more specific by considering a Coxeter graph of small type Γ with vertex set I, and its associated Artin-Tits group B, Artin-Tits monoid B^+, and set of positive roots Φ^+ (see section 1).

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Let V be a free $\mathfrak{R}$-module with basis $(e_\alpha)_{\alpha\in\Phi^+}$. The LK-representations of B^+ on V are linear representations $\psi : B^+ \to \mathscr{L}(V)$ parametrized by three elements $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ of the unitary group $\mathfrak{R}^\times$ of $\mathfrak{R}$, and by a family $(f_i)_{i\in I}$ of linear forms on V submitted to some extra conditions (see definition 12). For a fixed choice of $(\mathfrak{b},\mathfrak{c},\mathfrak{d}) \in (\mathfrak{R}^\times)^3$, the suitable families $(f_i)_{i\in I}$ — that we call *LK-families* — form a submodule $\mathscr{F}$ of the $\mathfrak{R}$-module $(V^\star)^I$. When the images of ψ are all invertible, then ψ induces a linear representation (also called LK) $\psi_{\mathrm{gr}} : B \to \mathrm{GL}(V)$ of the Artin-Tits group B ; this is precisely the case when the elements $f_i(e_{\alpha_i})$, $i \in I$, all belong to $\mathfrak{R}^\times$, and we denote by $\mathscr{F}_{\mathrm{gr}} \subseteq \mathscr{F} \setminus \{0\}$ the subset of LK-families that satisfy this additional condition.

Hence the classification of the LK-representations of B^+ reduces to the description of the $\mathfrak{R}$-module $\mathscr{F}$, and the question of the existence of LK-representations of B reduces to the question of the non-emptiness of $\mathscr{F}_{\mathrm{gr}}$. Note moreover that, when Γ is connected, the elements $f_i(e_{\alpha_i})$, $i \in I$, are necessarily all equal for an LK-family $(f_i)_{i\in I}$. The studies of [8, 12] essentially show that, in the connected and spherical cases, an LK-family is entirely determined by the common value $\mathfrak{f} \in \mathfrak{R}$ of the $f_i(e_{\alpha_i})$, $i \in I$, and that this common value can be chosen arbitrarily ; hence in these cases, $\mathscr{F}$ is isomorphic to $\mathfrak{R}$, via $(f_i)_{i\in I} \mapsto f_{i_0}(e_{\alpha_{i_0}})$ for some $i_0 \in I$, and $\mathscr{F}_{\mathrm{gr}}$ corresponds to $\mathfrak{R}^\times$ via this isomorphism (see subsection 3.2 below). This situation is partially generalized in [21] where it is shown that $\mathscr{F}_{\mathrm{gr}}$ is non-empty when Γ has *no triangle*, *i.e.* no subgraph of affine type $\tilde{A}_2$ (see subsection 3.3 below). As far as I know, the structure of $\mathscr{F}$ is not understood in general, and the question the non-emptiness of $\mathscr{F}_{\mathrm{gr}}$ is still open when Γ has a triangle.

Another topic on this subject is the question of the existence of similar faithful representations in the non-small cases. A first answer is provided by [12], where is constructed a faithful "twisted" LK-representation for an Artin-Tits group of type B_n, F_4 or G_2, using the fact that it appears as the subgroup of fixed elements under a graph automorphism, of an Artin-Tits group of type A_{2n-1}, E_6 or D_4 respectively.

The aim of this paper is to go further on those two questions.

We first investigate the structures of $\mathscr{F}$ and $\mathscr{F}_{\mathrm{gr}}$ when Γ is of affine and small type. We show that in these cases, an LK-family $(f_i)_{i\in I}$ is not determined by the common value $\mathfrak{f}$ of the $f_i(e_{\alpha_i})$, $i \in I$, but by an infinite family $(\mathfrak{f}_n)_{n\in\mathbb{N}} \in \mathfrak{R}^{\mathbb{N}}$ with $\mathfrak{f}_0 = \mathfrak{f}$, which can be chosen arbitrarily ; hence $\mathscr{F}$ is isomorphic to $\mathfrak{R}^{\mathbb{N}}$, via $(f_i)_{i\in I} \mapsto (\mathfrak{f}_n)_{n\in\mathbb{N}}$, and $\mathscr{F}_{\mathrm{gr}}$ corresponds to $\mathfrak{R}^\times \times \mathfrak{R}^{\mathbb{N}_{\geqslant 1}}$ via this isomorphism. In particular, this holds for $\tilde{A}_2$ and we thus get that $\mathscr{F}_{\mathrm{gr}}$ can be non-empty when Γ has triangles.

We then generalize the construction of (faithful) twisted LK-representations of [12] to any Artin-Tits monoid that appears as the submonoid of fixed elements, under a group of graph automorphisms, of an Artin-Tits monoid of small type. Note that our proof of faithfulness is different from the one of [12] as it does not use any case-by-case consideration : we show in general that the faithfulness criterion used in the small type cases also works in the twisted cases. In particular, starting with types A_{2n} and D_{n+1}, we get two new (faithful) LK-representations of the Artin-tits group of type B_n. By computing the formulas obtained for a twisted LK-representation when the considered group of graph automorphisms is of order two, we show that the three twisted LK-representations for the type B_n are pairwise non-equivalent, at least when $\mathfrak{R} = \mathbb{Z}[x^{\pm1}, y^{\pm1}]$ and for the main choices of parameters.

The paper is organized as follows.

We recall the basic results needed on Coxeter groups, root systems and Artin-Tits monoids and groups in the first section.

In the second section, we define the LK-representations over an arbitrary commutative ring $\mathfrak{R}$ of Artin-Tits monoids and groups of small type following [17, 8, 12, 21] (in subsection 2.1). We then generalize to our settings the faithfulness criterion on $\mathfrak{R}$ used in those articles and its short proof given in [14] (in subsection 2.2), and apply it in subsection 2.3.

In the third section, we investigate the module of LK-families $\mathscr{F}$ and its subset $\mathscr{F}_{\mathrm{gr}}$ for a fixed Coxeter graph of small type and a fixed choice of parameters $(\mathfrak{b}, \mathfrak{c}, \mathfrak{d}) \in (\mathfrak{R}^\times)^3$. The elements of $\mathscr{F}$ are characterized in subsection 3.1, and we recall the results of [8, 12] and [21] on $\mathscr{F}$ and its subset $\mathscr{F}_{\mathrm{gr}}$ in subsections 3.2 and 3.3 respectively. Subsection 3.4 is devoted to our study of the affine case.

Finally in section 4, we investigate the "twisted" LK-representations. We generalize the construction of [12] in subsection 4.1, and prove our "twisted" faithfulness criterion in subsection 4.2. We explicit the formulas of a twisted LK-representations when the considered group of graph automorphisms is of order two in subsection 4.3 and apply this in subsection 4.4 to compare the twisted LK-representations of type B_n. As a conclusion, we give in subsection 4.5 some limitations of our approach in the non-spherical cases compared with the spherical cases of [12].

1. Preliminaries

1.1. General notations and definitions.

In all this paper, the rings we consider will be unitary, with identity element denoted by 1 or Id. Let $\mathfrak{R}$ be a commutative ring. We denote by $\mathfrak{R}^\times$ the group of units of $\mathfrak{R}$. If V and V' are two $\mathfrak{R}$-modules, we denote by $\mathscr{L}(V, V')$ the $\mathfrak{R}$-module of linear maps from V to V'. If $V = V'$, we simply denote by $\mathscr{L}(V) = \mathscr{L}(V, V)$ the $\mathfrak{R}$-module of endomorphisms of V, by $\mathrm{GL}(V)$ the group of linear automorphisms of V and by $V^\star = \mathscr{L}(V, \mathfrak{R})$ the dual of V.

A monoid is a non-empty set endowed with an associative binary operation with an identity element. A monoid M is said to be *left cancellative* if for any a, b, $c \in M$, $ab = ac$ implies $b = c$. The notion of *right cancellativity* is defined symmetrically, and M is simply said to be *cancellative* when it is left and right cancellative. We denote by $\preccurlyeq$ the (left) *divisibility* in a monoid M, *i.e.* for a, $b \in M$, we write $b \preccurlyeq a$ if there exists $c \in M$ such that $a = bc$; this leads to the natural notions of (left) gcd's and (right) lcm's in M.

By *a linear representation* of a monoid M on an $\mathfrak{R}$-module V, we mean a monoid homomorphism $\varphi : M \to \mathscr{L}(V)$; for sake of brevity in this paper, we will often denote by φ_b the image $\varphi(b)$ of a given $b \in M$ by a linear representation φ. Two linear representations φ and φ' of a monoid M, on $\mathfrak{R}$-modules V and V' respectively, are said to be equivalent if there exists a linear isomorphism $\nu : V \to V'$ such that, for every $b \in M$, $\varphi'_b = \nu(\varphi_b)\nu^{-1}$.

1.2. Coxeter groups and Artin-Tits monoids and groups.

Let $\Gamma = (m_{i,j})_{i,j \in I}$ be a *Coxeter matrix*, *i.e.* with $m_{i,j} = m_{j,i} \in \mathbb{N}_{\geqslant 1} \cup \{\infty\}$ and $m_{i,j} = 1 \Leftrightarrow i = j$. We will always assume in this paper that I is finite ; this condition could be removed at a cost of some refinements in certain statements below (see [6, Ch. 11] for some of them), which are left to the reader.

As usual, we encode the data of Γ by its *Coxeter graph*, *i.e.* the graph with vertex set I, an edge between the vertices i and j if $m_{i,j} \geqslant 3$, and a label $m_{i,j}$ on that edge when $m_{i,j} \geqslant 4$. In the remainder of the paper, we will identify a Coxeter matrix with its Coxeter graph.

We denote by $W = W_\Gamma$ (resp. $B = B_\Gamma$, resp. $B^+ = B_\Gamma^+$) the *Coxeter group* (resp. *Artin-Tits group*, resp. *Artin-Tits monoid*) associated with Γ :

$$\begin{aligned} W &= \langle\, s_i,\, i \in I \mid \underbrace{s_i s_j s_i \cdots}_{m_{i,j}\text{ terms}} = \underbrace{s_j s_i s_j \cdots}_{m_{i,j}\text{ terms}} \text{ if } m_{i,j} \neq \infty, \text{ and } s_i^2 = 1 \,\rangle, \\ B &= \langle\, \boldsymbol{s}_i,\, i \in I \mid \underbrace{\boldsymbol{s}_i \boldsymbol{s}_j \boldsymbol{s}_i \cdots}_{m_{i,j}\text{ terms}} = \underbrace{\boldsymbol{s}_j \boldsymbol{s}_i \boldsymbol{s}_j \cdots}_{m_{i,j}\text{ terms}} \text{ if } m_{i,j} \neq \infty \,\rangle, \\ B^+ &= \langle\, \boldsymbol{s}_i,\, i \in I \mid \underbrace{\boldsymbol{s}_i \boldsymbol{s}_j \boldsymbol{s}_i \cdots}_{m_{i,j}\text{ terms}} = \underbrace{\boldsymbol{s}_j \boldsymbol{s}_i \boldsymbol{s}_j \cdots}_{m_{i,j}\text{ terms}} \text{ if } m_{i,j} \neq \infty \,\rangle^+. \end{aligned}$$

Note that there is no ambiguity in writing with the same symbols the generators of B and of B^+ since the canonical morphism $\iota : B^+ \to B$, given by the universal properties of the presentations, is injective [21], so B^+ can be identified with the submonoid of B generated by the $\boldsymbol{s}_i$, $i \in I$. We denote by ℓ the length function on B^+ relatively to its generating set $\{\boldsymbol{s}_i \mid i \in I\}$.

Let J be a subset of I. We denote by

- $\Gamma_J = (m_{i,j})_{i,j \in J}$ the submatrix of Γ of index set J,
- $W_J = \langle s_j,\, j \in J\rangle$ the subgroup of W generated by the s_j, $j \in J$,
- $B_J = \langle \boldsymbol{s}_j,\, j \in J\rangle$ the subgroup of B generated by the $\boldsymbol{s}_j$, $j \in J$,
- $B_J^+ = \langle \boldsymbol{s}_j,\, j \in J\rangle$ the submonoid of B^+ generated by the $\boldsymbol{s}_j$, $j \in J$.

It is known that W_J, (resp. B_J, resp. B_J^+) is the Coxeter group (resp. Artin-Tits group, resp. Artin-Tits monoid) associated with Γ_J (see [2, Ch. IV, n° 1.8, Thm. 2] for the Coxeter case, [22, Ch. II, Thm. 4.13] for the Artin-Tits group case, the Artin-Tits monoid case being obvious).

We say that J and Γ_J are *spherical* if W_J is finite, or, equivalently, if the elements $\boldsymbol{s}_j$, $j \in J$, have a common (right) multiple in B^+. In that case, the elements $\boldsymbol{s}_j$, $j \in J$, have a unique (right) lcm in B^+, denoted by Δ_J and called *the Garside element* of B_J^+. Moreover, the group B_J is then the group of (left) fractions of B_J^+, *i.e.* every $b \in B_J$ can be written $b = b'^{-1}b''$ with b', $b'' \in B_J^+$ (see [3, Props. 4.1, 5.5 and Thm. 5.6]).

For $b \in B^+$, we set $I(b) = \{i \in I \mid \boldsymbol{s}_i \preccurlyeq b\}$. In view of what has just been said, $I(b)$ is a spherical subset of I.

Let us conclude this subsection by the following easy, but fundamental, lemma :

Lemma 1. *Consider a monoid homomorphism $\psi : B^+ \to G$, where G is a group. Then ψ extends to a group homomorphism $\psi_{\mathrm{gr}} : B \to G$ such that $\psi = \psi_{\mathrm{gr}} \circ \iota$.*

Moreover if Γ is spherical and if ψ is injective, then ψ_{gr} is injective.

Proof. The universal property of B gives the first part. For the second, take $b \in \ker(\psi_{\mathrm{gr}})$ and consider a decomposition $b = b'^{-1}b''$ with b', $b'' \in B^+$. Then $\psi_{\mathrm{gr}}(b) = 1$ means $\psi(b') = \psi_{\mathrm{gr}}(b') = \psi_{\mathrm{gr}}(b'') = \psi(b'')$, whence $b' = b''$ by injectivity of ψ and hence $b = 1$. $\square$

Note that if one is able to construct an injective morphism $\psi : B^+ \to G$ where G is a group, then one gets that the canonical morphism ι is injective ; this is the idea of [21]. In this paper, we will be interested in representations ψ of B^+ in some linear group $\mathrm{GL}(V)$, hence proving their faithfulness will prove at the same time the faithfulness of the corresponding linear representation $\psi_{\mathrm{gr}} : B \to \mathrm{GL}(V)$ when Γ is spherical.

1.3. **Standard root systems.**

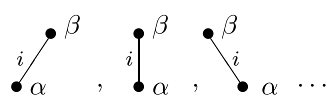
Let $\Gamma = (m_{i,j})_{i,j\in I}$ be a Coxeter matrix. Details on the notions introduced here can be found in [11].

Let $E = \oplus_{i\in I}\mathbb{R}\alpha_i$ be a $\mathbb{R}$-vector space with basis $(\alpha_i)_{i\in I}$ indexed by I. We endow E with a symmetric bilinear form $(\,.\,|\,.\,) = (\,.\,|\,.\,)_\Gamma$ given on the basis $(\alpha_i)_{i\in I}$ by $(\alpha_i|\alpha_j) = -2\cos\left(\frac{\pi}{m_{i,j}}\right)$. The Coxeter group $W = W_\Gamma$ acts on E via $s_i(\beta) = \beta - (\beta|\alpha_i)\alpha_i$.

The (*standard*) *root system* associated with Γ is by definition the set $\Phi = \Phi_\Gamma = \{w(\alpha_i) \mid w \in W, i \in I\}$. It is well-known that $\Phi = \Phi^+ \sqcup \Phi^-$, where $\Phi^+ = \Phi \bigcap (\oplus_{i\in I}\mathbb{R}^+\alpha_i)$ and $\Phi^- = -\Phi^+$. For $\alpha = \sum_{i\in I}\lambda_i\alpha_i \in \Phi$ we call *support* of α the set $\mathrm{Supp}(\alpha) = \{i \in I \mid \lambda_i \neq 0\}$.

We will always represent a subset Ψ of Φ^+ by a graph with vertex set Ψ and an edge labeled i between two vertices α and β if $\alpha = s_i(\beta)$. For example, the situation where β is fixed by s_i will be drawn by a loop β i .

Such a graph is naturally $\mathbb{N}$-graded via the *depth* function on Φ^+, where the depth of a root $\alpha \in \Phi^+$ is by definition $\mathrm{dp}(\alpha) = \min\{l(w) \mid w \in W, w(\alpha) \in \Phi^-\}$. Contrary to what suggests this terminology, in all the graphs that we will draw, we chose to place a root of great depth *above* a root of small depth ; so drawings like the following ones (with β *above* α), will all mean that $\beta = s_i(\alpha)$ (or equivalently $\alpha = s_i(\beta)$) and $\mathrm{dp}(\beta) > \mathrm{dp}(\alpha)$:



Lemma 2 ([4, Lem 1.7])**.** *Let $i \in I$ and $\alpha \in \Phi^+ \setminus \{\alpha_i\}$. Then*

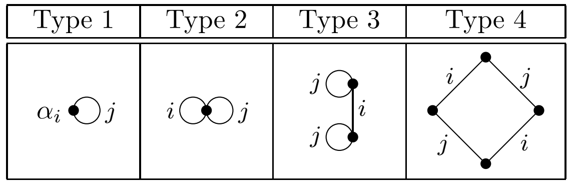
$$\mathrm{dp}(s_i(\alpha)) = \begin{cases} \mathrm{dp}(\alpha) - 1 & \text{if } (\alpha|\alpha_i) > 0, \\ \mathrm{dp}(\alpha) & \text{if } (\alpha|\alpha_i) = 0, \\ \mathrm{dp}(\alpha) + 1 & \text{if } (\alpha|\alpha_i) < 0, \end{cases}$$

In the remainder of the paper, we will often consider subsets of Φ^+ of the form $\{w(\alpha) \mid w \in W_{\{i,j\}}\} \bigcap \Phi^+$, for $\alpha \in \Phi^+$ and $i, j \in I$ with $m_{i,j} = 2$ or 3, so the following definition and remark will be useful :

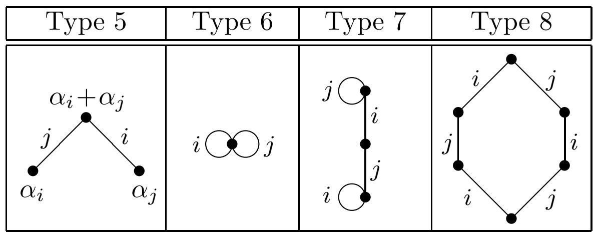
Definition 3. Let $\alpha \in \Phi^+$ and $J \subseteq I$. We call *J-mesh* of α, or simply *mesh*, the set $[\alpha]_J := \{w(\alpha) \mid w \in W_J\} \bigcap \Phi^+$. This terminology is inspired by personal communications with Hée.

Remark 4. Let $\alpha \in \Phi^+$ and $i, j \in I$ with $m_{i,j} = 2$ or 3. Then, up to exchanging i and j, the graph of the mesh $[\alpha]_{\{i,j\}}$ is one of the following :

- if $m_{i,j} = 2$:



- if $m_{i,j} = 3$:



Let J be a subset of I. We denote by Φ_J the subset $\{w(\alpha_j) \mid w \in W_J,\ j \in J\}$ of Φ. It is clear that Φ_J is the root system associated with Γ_J in $\oplus_{j\in J}\mathbb{R}\alpha_j$.

1.4. **Graph automorphisms.**

We call *automorphism* of a Coxeter matrix $\Gamma = (m_{i,j})_{i,j\in I}$ every permutation g of I such that $m_{g(i),g(j)} = m_{i,j}$ for all $i,\ j \in I$, and we denote by $\mathrm{Aut}(\Gamma)$ the group the constitute.

Any automorphism of Γ clearly acts by automorphisms on W, B and B^+ by permuting the corresponding generating set. If G is a subgroup of $\mathrm{Aut}(\Gamma)$, we denote by W^G, B^G and $(B^+)^G$ the corresponding subset of fixed points under the action of the elements of G. It is known that W^G (resp. $(B^+)^G$) is a Coxeter group (resp. Artin-Tits monoid) associated with a certain Coxeter graph Γ' easily deduced from Γ, and the analogue holds for B^G when Γ is spherical, or more generally of *FC-type* (see [13, 20] for the Coxeter case, [19, 9, 10, 7] for the Artin-Tits case). Note that the standard generator of $(B^+)^G$ are the Garside elements Δ_J of B_J^+, for J running through the spherical orbits of I under G.

Similarly, any automorphism g of Γ acts by a linear automorphism on $E = \oplus_{i\in I}\mathbb{R}\alpha_i$ by permuting the basis $(\alpha_i)_{i\in I}$. This action stabilizes Φ and Φ^+, and the induced action on those sets is given by $w(\alpha_i) \mapsto (g(w))(\alpha_{g(i)})$.

2. LK-representations

In subsection 2.1 below, we define the Lawrence-Krammer representations, over an arbitrary (unitary) commutative ring $\mathfrak{R}$, of the Artin-Tits monoids an groups of small type. The definition is inspired by the ones of [17, 8, 12, 21], where $\mathfrak{R}$ is chosen to be $\mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ (*cf.* section 2.3 below).

In subsection 2.2, we extend to our settings the faithfulness criterion of [17, 8, 12, 21] and prove it following [14]. We apply that criterion in subsection 2.3.

From now on, we assume that $\Gamma = (m_{i,j})_{i,j\in I}$ is a Coxeter matrix of *small* type, *i.e.* with $m_{i,j} \in \{1, 2, 3\}$ for all $i,\ j \in I$.

2.1. **Definition.**

Let $\mathfrak{R}$ be a commutative ring and V be a free $\mathfrak{R}$-module with basis $(e_\alpha)_{\alpha\in\Phi^+}$ indexed by Φ^+.

Notation 5. For $f \in V^\star$ and $e \in V$, we denote by $f \boxtimes e$ the element of $\mathscr{L}(V)$ given by $(f\boxtimes e)(v) = f(v)e$ for every $v \in V$.

Remark 6. Consider $\varphi \in \mathscr{L}(V)$, f, $f' \in V^\star$ and e, $e' \in V$. Then :

(i) $\varphi(f\boxtimes e) = f\boxtimes\varphi(e)$,
(ii) $(f\boxtimes e)\varphi = (f\varphi)\boxtimes e$,
(iii) $(f\boxtimes e)(f'\boxtimes e') = f(e')(f'\boxtimes e)$.

Definition 7. Fix $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}) \in \mathfrak{R}^4$ and a family of linear forms $(f_i)_{i\in I} \in (V^\star)^I$. For $i \in I$, we denote by $\mathfrak{f}_{i,\alpha}$ the element $f_i(e_\alpha)$, for $\alpha \in \Phi^+$, and by

- φ_i the endomorphism of V given on the basis $(e_\alpha)_{\alpha\in\Phi^+}$ by

$$\begin{cases} \varphi_i(e_\alpha) = 0 & \text{if} \quad \alpha = \alpha_i, \\ \varphi_i(e_\alpha) = \mathfrak{d}e_\alpha & \text{if} \quad \overset{\alpha}{\bullet}\!\!\bigcirc\, i, \\ \begin{cases} \varphi_i(e_\beta) = \mathfrak{b}e_\alpha \\ \varphi_i(e_\alpha) = \mathfrak{a}e_\alpha + \mathfrak{c}e_\beta \end{cases} & \text{if} \quad \begin{array}{c} \beta\bullet \\ \alpha\bullet \end{array} i \quad \text{in } \Phi^+. \end{cases}$$

- ψ_i the endomorphism of V given by $\psi_i = \varphi_i + f_i\boxtimes e_{\alpha_i}$.

Remark 8. If one fixes an arrangement of the basis $(e_\alpha)_{\alpha\in\Phi^+}$ so that e_{α_i} is the leftmost element and e_β is the right successor of e_α whenever $\beta = s_i(\alpha)$ with $\mathrm{dp}(\beta) > \mathrm{dp}(\alpha)$, then the matrix of φ_i in this basis is block diagonal, with blocks

$$\begin{cases} \begin{array}{c} e_\alpha \\ \begin{pmatrix} 0 \end{pmatrix} \end{array} & \text{if} \quad \alpha = \alpha_i, \\ \begin{array}{c} e_\alpha \\ \begin{pmatrix} \mathfrak{d} \end{pmatrix} \end{array} & \text{if} \quad \overset{\alpha}{\bullet}\!\!\bigcirc\, i, \\ \begin{array}{cc} e_\alpha & e_\beta \\ \multicolumn{2}{c}{\begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & 0 \end{pmatrix}} \end{array} & \text{if} \quad \begin{array}{c} \beta\bullet \\ \alpha\bullet \end{array} i \ . \end{cases}$$

And the matrix of ψ_i is the same except that the first row (the one of index α_i), which is zero in φ_i, if replaced by the row $(\mathfrak{f}_{i,\alpha})_{\alpha\in\Phi^+} = (f_i(e_\alpha))_{\alpha\in\Phi^+}$.

Let us now exhibit conditions on $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ and $(f_i)_{i\in I}$ so that the map $\boldsymbol{s}_i \mapsto \psi_i$ extends to a linear representation ψ of B^+ into $\mathscr{L}(V)$ or $\mathrm{GL}(V)$.

Lemma 9. *The map ψ_i is invertible if and only if $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ and $\mathfrak{f}_{i,\alpha_i}$ belong to $\mathfrak{R}^\times$, in which case the inverse of ψ_i is given by*

$$\begin{cases} \psi_i^{-1}(e_\alpha) = \dfrac{1}{\mathfrak{f}_{i,\alpha_i}} e_{\alpha_i} & \textit{if} \quad \alpha = \alpha_i, \\ \psi_i^{-1}(e_\alpha) = \dfrac{1}{\mathfrak{d}}\Big(e_\alpha - \dfrac{\mathfrak{f}_{i,\alpha}}{\mathfrak{f}_{i,\alpha_i}} e_{\alpha_i}\Big) & \textit{if} \quad \overset{\alpha}{\bullet}\!\!\bigcirc\, i, \\ \begin{cases} \psi_i^{-1}(e_\beta) = \dfrac{1}{\mathfrak{c}}\Big(e_\alpha - \dfrac{\mathfrak{a}}{\mathfrak{b}} e_\beta + \dfrac{\mathfrak{a}\mathfrak{f}_{i,\beta} - \mathfrak{b}\mathfrak{f}_{i,\alpha}}{\mathfrak{b}\mathfrak{f}_{i,\alpha_i}} e_{\alpha_i}\Big) \\ \psi_i^{-1}(e_\alpha) = \dfrac{1}{\mathfrak{b}}\Big(e_\beta - \dfrac{\mathfrak{f}_{i,\beta}}{\mathfrak{f}_{i,\alpha_i}} e_{\alpha_i}\Big) \end{cases} & \textit{if} \quad \begin{array}{c} \beta\bullet \\ \alpha\bullet \end{array} i \quad \textit{in } \Phi^+. \end{cases}$$

Proof. Straightforward computations. □

Lemma 10. *Consider i, $j \in I$ with $i \neq j$.*

(i) *If $m_{i,j} = 2$, then $\varphi_i\varphi_j = \varphi_j\varphi_i$.*
(ii) *If $m_{i,j} = 3$ and if $\mathfrak{a}\big(\mathfrak{d}(\mathfrak{a}-\mathfrak{d})+\mathfrak{b}\mathfrak{c}\big) = 0$, then $\varphi_i\varphi_j\varphi_i = \varphi_j\varphi_i\varphi_j$.*

Proof. For every $\alpha \in \Phi^+$, the linear maps φ_i and φ_j stabilize the submodule of V generated by the elements e_β for β running through the $\{i,j\}$-mesh $[\alpha]_{i,j}$ of α. The results then follow from the direct computations of the matrices of the restrictions of $\varphi_i\varphi_j$ and $\varphi_i\varphi_j\varphi_i$ to those submodules (of dimension 1, 2, 3, 4 or 6 in view of remark 4). Note that the only case where the condition $\mathfrak{a}\big(\mathfrak{d}(\mathfrak{a}-\mathfrak{d})+\mathfrak{b}\mathfrak{c}\big) = 0$ is needed is the case of a mesh of type 7 in the nomenclature of remark 4. □

Lemma 11. *Consider i, $j \in I$ with $i \neq j$, and assume that $\mathfrak{d}(\mathfrak{a}-\mathfrak{d})+\mathfrak{b}\mathfrak{c} = 0$ and $f_i(\alpha_j) = f_j(\alpha_i) = 0$.*

(i) *If $m_{i,j} = 2$, then $\psi_i\psi_j = \psi_j\psi_i$ if and only if $f_i\varphi_j = \mathfrak{d}f_i$ and $f_j\varphi_i = \mathfrak{d}f_j$.*
(ii) *If $m_{i,j} = 3$, then $f_i\varphi_j = f_j\varphi_i$ implies $\psi_i\psi_j\psi_i = \psi_j\psi_i\psi_j$, and the converse is true if $\mathfrak{c} \in \mathfrak{R}^\times$.*

Proof. Note that, since $f_i(\alpha_j) = 0$, we get, by using the formulas of remark 6 : $(f_i \boxtimes v)(f \boxtimes e_{\alpha_j}) = 0$ for every $(v, f) \in V \times V^\star$ (and similarly if we exchange i and j), and hence

- $\psi_i\psi_j = \varphi_i\varphi_j + f_j \boxtimes \varphi_i(e_{\alpha_j}) + (f_i\varphi_j) \boxtimes e_{\alpha_i}$, and
- $\psi_i\psi_j\psi_i = \varphi_i\varphi_j\varphi_i + f_i \boxtimes \varphi_i\varphi_j(e_{\alpha_i}) + (f_j\varphi_i) \boxtimes \varphi_i(e_{\alpha_j}) + \big(f_i\varphi_j\varphi_i + (f_i\varphi_j)(e_{\alpha_i})f_i\big) \boxtimes e_{\alpha_i}$.

If $m_{i,j} = 2$, then $\varphi_i(e_{\alpha_j}) = \mathfrak{d}e_{\alpha_j}$, thus we get, by symmetry in i and j and by case (i) of the previous lemma :

$$\psi_i\psi_j - \psi_j\psi_i = (f_i\varphi_j - \mathfrak{d}f_i) \boxtimes e_{\alpha_i} - (f_j\varphi_i - \mathfrak{d}f_j) \boxtimes e_{\alpha_j}.$$

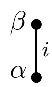
This establishes (i). If $m_{i,j} = 3$, then $\varphi_i(e_{\alpha_j}) = \mathfrak{a}e_{\alpha_j} + \mathfrak{c}e_{\alpha_i+\alpha_j}$, $\varphi_i\varphi_j(e_{\alpha_i}) = \mathfrak{b}\mathfrak{c}e_{\alpha_j}$, thus we get, by symmetry in i and j and by case (ii) of the previous lemma :

$$\begin{aligned}\psi_i\psi_j\psi_i - \psi_j\psi_i\psi_j &= \mathfrak{c}(f_j\varphi_i - f_i\varphi_j) \boxtimes e_{\alpha_i+\alpha_j} \\ &\quad + \big(f_j\varphi_i(\varphi_j - \mathfrak{a}\,\mathrm{Id}) + (f_j\varphi_i)(e_{\alpha_j})f_j - \mathfrak{b}\mathfrak{c}f_i\big) \boxtimes e_{\alpha_j} \\ &\quad - \underbrace{\big(f_i\varphi_j(\varphi_i - \mathfrak{a}\,\mathrm{Id}) + (f_i\varphi_j)(e_{\alpha_i})f_i - \mathfrak{b}\mathfrak{c}f_j\big)}_{F_{i,j}} \boxtimes e_{\alpha_i}.\end{aligned}$$

The second part of (ii) is now clear, and to show the direct implication, we have to show that $f_i\varphi_j = f_j\varphi_i$ implies $F_{i,j} = 0$ (this will give $F_{j,i} = 0$ by symmetry and hence $\psi_i\psi_j\psi_i = \psi_j\psi_i\psi_j$). But since $\varphi_i(e_{\alpha_i}) = 0$, we have

$$F_{i,j} = (f_i\varphi_j - f_j\varphi_i)(\varphi_i - \mathfrak{a}\,\mathrm{Id}) + (f_i\varphi_j - f_j\varphi_i)(e_{\alpha_i})f_i + f_j\big(\varphi_i^2 - \mathfrak{a}\varphi_i - \mathfrak{b}\mathfrak{c}\,\mathrm{Id}\big),$$

and the linear form $f_j\big(\varphi_i^2 - \mathfrak{a}\varphi_i - \mathfrak{b}\mathfrak{c}\,\mathrm{Id}\big)$ is the zero form, since it is zero on

- e_{α_i} since $\varphi_i(e_{\alpha_i}) = 0$ and $f_j(e_{\alpha_i}) = 0$,
- e_α if  since then $\varphi_i(e_\alpha) = \mathfrak{d}e_\alpha$ and $\mathfrak{d}^2 - \mathfrak{a}\mathfrak{d} - \mathfrak{b}\mathfrak{c} = 0$,
- e_α and e_β if  in Φ^+, since $X^2 - \mathfrak{a}X - \mathfrak{b}\mathfrak{c}$ is the characteristic polynomial of the restriction of φ_i on $\mathfrak{R}e_\alpha \oplus \mathfrak{R}e_\beta$ (see remark 8).

Whence the result. □

Definition 12. Fix $(\mathfrak{b},\mathfrak{c},\mathfrak{d}) \in (\mathfrak{R}^\times)^3$, set $\mathfrak{a} = \mathfrak{d} - \dfrac{\mathfrak{b}\mathfrak{c}}{\mathfrak{d}}$, and consider the linear maps $\varphi_i \in \mathscr{L}(V)$, $i \in I$, as in definition 7. We say that a family $(f_i)_{i\in I} \in (V^\star)^I$ is an *LK-family* (*relatively to* $(\mathfrak{b},\mathfrak{c},\mathfrak{d})$) if it satisfies the following properties :

(i) for i, $j \in I$ with $i \neq j$, $f_i(e_{\alpha_j}) = 0$,
(ii) for i, $j \in I$ with $m_{i,j} = 2$, $f_i\varphi_j = \mathfrak{d}f_i$,
(iii) for i, $j \in I$ with $m_{i,j} = 3$, $f_i\varphi_j = f_j\varphi_i$.

We denote by $\mathscr{F} = \mathscr{F}_{(\mathfrak{b},\mathfrak{c},\mathfrak{d})}$ the set of LK-families relatively to $(\mathfrak{b},\mathfrak{c},\mathfrak{d})$. This is clearly a submodule of the $\mathfrak{R}$-module $(V^\star)^I$. We denote by $\mathscr{F}_{\mathrm{gr}}$ the subset of $\mathscr{F}$ composed of the LK-families for which $f_i(e_{\alpha_i}) \in \mathfrak{R}^\times$ for every $i \in I$.

In view of lemma 11 above, for every LK-family $(f_i)_{i\in I}$, the map $\boldsymbol{s}_i \mapsto \psi_i = \varphi_i + f_i \boxtimes e_{\alpha_i}$ extends to a linear representation $\psi = \psi_{(\mathfrak{b},\mathfrak{c},\mathfrak{d}),(f_i)_{i\in I}} : B^+ \to \mathscr{L}(V)$. Moreover, if $(f_i)_{i\in I} \in \mathscr{F}_{\mathrm{gr}}$, then in view of lemma 9, the images of ψ are invertible, and hence $\psi : B^+ \to \mathrm{GL}(V)$ induces a linear representation $\psi_{\mathrm{gr}} : B \to \mathrm{GL}(V)$.

We call *Lawrence-Krammer representation* — *LK-representation* for short — the representation ψ of B^+ and, when appropriate, the representation ψ_{gr} of B.

Remark 13. The assumption on $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ to be units of $\mathfrak{R}$ is not needed to define the LK-representations of B^+ and for the faithfulness criterion of the following subsection. We included it in the definition since we are mainly interested in LK-representations of B^+ that extends to LK-representations of B, and since it will be of importance in our general study of LK-families in section 3 below.

2.2. **Faithfulness criterion.**

The key argument in [16, 17, 8, 12, 21] is that the LK-representation ψ they consider is faithful. The faithfulness criterion used each time can be summarized as follows (where $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $(f_i)_{i\in I}$ and ψ are as in definition 12) :

Theorem 14. *Assume that the following two conditions are satisfied :*

(i) $\mathrm{Im}(\psi)$ *is a left cancellative submonoid of* $\mathscr{L}(V)$,
(ii) *there exists a totally ordered commutative ring* $\mathfrak{R}_0$ *and a ring homomorphism* $\mathfrak{R} \to \mathfrak{R}_0$, $\mathfrak{x} \mapsto \overline{\mathfrak{x}}$, *such that* $\overline{\mathfrak{a}}$, $\overline{\mathfrak{b}}$, $\overline{\mathfrak{c}}$, $\overline{\mathfrak{d}}$ *are positive and* $\overline{\mathfrak{f}_{i,\alpha}} = 0$ *for every* $(i,\alpha) \in I \times \Phi^+$.

Then the LK-representation ψ *is faithful.*

In the remainder of this subsection, we sketch the (much easier) proof of this criterion obtained by Hée in [14]. It does not involve any consideration on closed sets of positive roots, nor on the maximal simple (left) divisor of an element of B^+, and rely only on the two following (elementary) lemmas and a look at the defining formulas of ψ.

Lemma 15 ([14, Prop. 1])**.** *Let* $\rho : B^+ \to M$ *be a monoid homomorphism where* M *is left cancellative. If* ρ *satisfies* $\rho(b) = \rho(b') \Rightarrow I(b) = I(b')$ *for all* b, $b' \in B^+$, *then it is injective.*

Proof. Under those assumptions, one can show, by induction on $\ell(b)$, that $\rho(b) = \rho(b')$ implies $b = b'$. (See lemma 49 below for a twisted version of that result.) $\square$

Notation 16. We denote by $\mathrm{Bin}(\Omega)$ the monoid of binary relations on a set Ω, where the product RR' of two binary relations R and R' is defined on Ω by $\beta RR'\alpha \Leftrightarrow \exists\,\gamma \in \Omega$ such that $\beta R\gamma$ and $\gamma R'\alpha$.

For $R \in \mathrm{Bin}(\Omega)$ and $\Psi \subseteq \Omega$, we denote by $R(\Psi)$ the set $\{\beta \in \Omega \mid \exists\,\alpha \in \Psi,\ \beta R\alpha\}$.

We will use again the following lemma in the proof of our faithfulness criterion in subsection 4.2 below. So, for completeness, we detail its proof here.

Lemma 17 ([14, Prop. 2])**.** *Let $B^+ \to \mathrm{Bin}(\Omega)$, $b \mapsto R_b$, be a monoid homomorphism, and let $(\alpha_i)_{i\in I}$ be a family of elements of Ω such that*

(i) $\alpha_i \notin R_{\boldsymbol{s}_i}(\Omega)$,
(ii) *if $i \neq j$, then $\alpha_i R_{\boldsymbol{s}_j} \alpha_i$,*
(iii) *if $m_{i,j} = 3$, then $\alpha_i R_{\boldsymbol{s}_j} R_{\boldsymbol{s}_i} \alpha_j$.*

Then for every $b \in B^+$, we have $\boldsymbol{s}_i \preccurlyeq b \Leftrightarrow \alpha_i \notin R_b(\Omega)$. In particular, for $b, b' \in B^+$, we get $R_b(\Omega) = R_{b'}(\Omega) \Rightarrow I(b) = I(b')$.

Proof. Since the map $b \to R_b$ is a monoid homomorphism, we get that $b' \preccurlyeq b$ implies $R_b(\Omega) \subseteq R_{b'}(\Omega)$ for every $b, b' \in B^+$. Thanks to property (i), this shows that $\boldsymbol{s}_i \preccurlyeq b \Rightarrow \alpha_i \notin R_b(\Omega)$. For the converse, assume that $\boldsymbol{s}_i \not\preccurlyeq b$ and let us prove by induction on $\ell(b)$ that $\alpha_i \in R_b(\Omega)$. If $b = 1$, then R_b is the equality relation and hence $\alpha_i \in R_b(\Omega)$. If $\ell(b) > 0$, fix $j \in I$ such that $\boldsymbol{s}_j \preccurlyeq b$ (hence $j \neq i$) and denote by b_1 the element of B^+ such that $b = \boldsymbol{s}_j b_1$; if $\boldsymbol{s}_i \not\preccurlyeq b_1$, then $\alpha_i \in R_{b_1}(\Omega)$ by induction and hence $\alpha_i \in R_b(\Omega)$ thanks to property (ii) ; if $\boldsymbol{s}_i \preccurlyeq b_1$, then $b = \boldsymbol{s}_j \boldsymbol{s}_i b_2$ for some $b_2 \in B^+$, and since $\boldsymbol{s}_i \not\preccurlyeq b$, we necessarily have $m_{i,j} \neq 2$ (hence $m_{i,j} = 3$) and $\boldsymbol{s}_j \not\preccurlyeq b_2$, so $\alpha_j \in R_{b_2}(\Omega)$ by induction and $\alpha_i \in R_b(\Omega)$ thanks to property (iii). ☐

Remark 18. Let $\mathfrak{R}_0$ be a totally ordered commutative ring and let V_0 be a free $\mathfrak{R}_0$-module with basis $(e_\alpha)_{\alpha\in\Omega}$. We denote by $\mathfrak{R}_0^+$ the set (semiring) of non-negative elements of $\mathfrak{R}_0$ and by $\mathscr{L}^+(V_0)$ the submonoid of $\mathscr{L}(V_0)$ composed of the linear maps $\varphi : V_0 \to V_0$ such that $\varphi(e_\alpha) \in \oplus_{\beta\in\Omega}\mathfrak{R}_0^+ e_\beta$ for all $\alpha \in \Omega$.

Then there is a monoid homomorphism $\mathscr{L}^+(V_0) \to \mathrm{Bin}(\Omega)$, $\varphi \mapsto R_\varphi$, where R_φ is given by $\beta R_\varphi \alpha \Leftrightarrow$ the coefficient of e_β in $\varphi(e_\alpha)$ is positive.

Now assume that we are in the situation of condition (ii) of theorem 14.

Definition 19 ([14, 4.3])**.** If we denote by V_0 the free $\mathfrak{R}_0$-module with basis $(e_\alpha)_{\alpha\in\Phi^+}$, then the ring homomorphism $\mathfrak{R} \to \mathfrak{R}_0$, $\mathfrak{x} \mapsto \bar{\mathfrak{x}}$, naturally induces a monoid homomorphism $\mathscr{L}(V) \to \mathscr{L}(V_0)$, $\varphi \mapsto \overline{\varphi}$, which sends $\mathrm{Im}(\psi)$ into $\mathscr{L}^+(V_0)$ by assumption on the parameters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ and $\mathfrak{f}_{i,\alpha}$, $(i,\alpha) \in I \times \Phi^+$. If we compose again by the monoid homomorphism $\mathscr{L}^+(V_0) \to \mathrm{Bin}(\Phi^+)$, $\varphi \mapsto R_\varphi$, of remark 18, we thus get a monoid homomorphism

$$B^+ \to \mathrm{Bin}(\Phi^+),\ b \mapsto R_b = R_{\overline{\psi_b}}$$

A quick look at the formulas of definition 7 then easily gives the following :

Lemma 20 ([14, 4.3 and 4.4])**.** *Under condition (ii) of theorem 14 and with the notations introduced in definition 19 above, the morphism $B^+ \to \mathrm{Bin}(\Phi^+)$, $b \mapsto R_b$, satisfies the following properties :*

(i) $\alpha_i \notin R_{\boldsymbol{s}_i}(\Phi^+)$,
(ii) *if $i \neq j$, then $\alpha_i R_{\boldsymbol{s}_j} \alpha_i$,*
(iii) *if $m_{i,j} = 3$, then $\alpha_i R_{\boldsymbol{s}_j} R_{\boldsymbol{s}_i} \alpha_j$.*

And a combination of the three previous lemmas easily gives theorem 14.

2.3. Application of the faithfulness criterion.

The typical situation where theorem 14 applies is, for any totally ordered commutative ring $\mathfrak{R}_0$ (for example any subring of $\mathbb{R}$) :

- $\mathfrak{R} = \mathfrak{R}_0[x]$ where x is an indeterminate, and the evaluation at $x = 0$ for the morphism $\mathfrak{R} \to \mathfrak{R}_0$,
- $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ positive units of $\mathfrak{R}_0$ with $\mathfrak{bc} < \mathfrak{d}^2$,
- an LK-family $(f_i)_{i\in I}$ with $\mathrm{Im}(f_i) \subseteq x\mathfrak{R}$ and $f_i(e_{\alpha_i}) \neq 0$ for all $i \in I$.

Indeed, in that situation, condition (ii) of theorem 14 is clearly satisfied. To see condition (i), note that, since the elements $f_i(e_{\alpha_i})$, $i \in I$, are non-zero, they become units of some appropriate overring $\mathfrak{R}'$ of $\mathfrak{R}$ (for example its field of fractions $\mathbb{K}(x)$, where $\mathbb{K}$ is the field of fractions of $\mathfrak{R}_0$). So if we denote by V' the free $\mathfrak{R}'$-module with basis $(e_\alpha)_{\alpha\in\Phi^+}$, then $\mathrm{Im}(\psi)$ is included in $\mathscr{L}(V) \cap \mathrm{GL}(V')$ and hence is cancellative.

Moreover, the faithful LK-representation ψ then induces an LK-representation $\psi_{\mathrm{gr}} : B \to \mathrm{GL}(V')$, which is faithful when Γ is spherical (in view of lemma 1).

Hence to construct faithful LK-representations of B^+ — and of B when Γ is spherical — it suffices to construct LK-families over $x\mathfrak{R}_0[x]$ with non-zero elements $f_i(e_{\alpha_i})$ for $i \in I$. This will be done, for any Coxeter graph of small type with no triangle, and for the triangle graph $\tilde{A}_2$, in subsection 3.5 below.

Example 21. Following [17, 8, 12, 21], one can choose $\mathfrak{R}_0 = \mathbb{Z}[y^{\pm 1}]$ for some $y \in \mathbb{R}_+^\star \setminus \{1\}$, and $(\mathfrak{b}, \mathfrak{c}, \mathfrak{d}) = (y^p, y^q, y^r)$ with p, q, $r \in \mathbb{Z}$ such that $2r < p + q$ (resp. $2r > p + q$) if $0 < y < 1$ (resp. $y > 1$).

In [17, 8, 12, 21], the authors choose $0 < y < 1$, $\mathfrak{d} = 1$ and $(\mathfrak{b}, \mathfrak{c}) = (y, y)$ (in [8]), $(1, y)$ (in [12]) or $(y, 1)$ (in [21]). Note that the situation in [17] is slightly out of our settings since the value of $(\mathfrak{b}, \mathfrak{c})$ varies for φ_i (between $(y, 1)$ and $(1, y)$), depending on the considered $\{i\}$-mesh of cardinality two. The authors construct LK-families $(f_i)_{i\in I}$ over $x\mathbb{Z}[y]$ with elements $f_i(e_{\alpha_i})$, $i \in I$, all equal to xy^4 (in [8]) or xy^2 (in [17, 12, 21]), hence the overring $\mathfrak{R}' = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ is appropriate in the discussion above.

3. On LK-families

We study in this section the module of LK-families $\mathscr{F}$ and its subset $\mathscr{F}_{\mathrm{gr}}$ for a given choice of parameters $(\mathfrak{b}, \mathfrak{c}, \mathfrak{d}) \in (\mathfrak{R}^\times)^3$, as defined in definition 12.

In subsection 3.1, we characterize the LK-families in terms of relations between the elements $\mathfrak{f}_{i,\alpha} = f_i(e_\alpha) \in \mathfrak{R}$, for every $(i, \alpha) \in I \times \Phi^+$. This characterization generalizes some computations of [8, 12, 21], and we recall in subsections 3.2 and 3.3 below their results on those families : in our settings, it is shown in [8, 12] that, when Γ is spherical, there exists an isomorphism from $\mathscr{F}$ onto $\mathfrak{R}$ which sends $\mathscr{F}_{\mathrm{gr}}$ onto $\mathfrak{R}^\times$, an it is shown in [21] that $\mathscr{F}_{\mathrm{gr}}$ is not trivial when Γ has no triangle.

The aim of subsection 3.4 is to explicit the structure of $\mathscr{F}$ and $\mathscr{F}_{\mathrm{gr}}$ when Γ is affine : we show that there exists an isomorphism from $\mathscr{F}$ onto $\mathfrak{R}^{\mathbb{N}}$ which sends $\mathscr{F}_{\mathrm{gr}}$ onto $\mathfrak{R}^\times \times \mathfrak{R}^{\mathbb{N}_{\geqslant 1}}$ (see theorem 35). In particular, this result holds for the affine type $\tilde{A}_2$ and hence gives the firsts examples of LK-representations of an Artin-Tits group whose type has triangles.

In all this section, we fix a Coxeter matrix of small type $\Gamma = (m_{i,j})_{i,j\in I}$, a commutative ring $\mathfrak{R}$, a triple $(\mathfrak{b}, \mathfrak{c}, \mathfrak{d}) \in (\mathfrak{R}^\times)^3$ and we set $\mathfrak{a} = \mathfrak{d} - \frac{\mathfrak{bc}}{\mathfrak{d}}$. We denote by V the free $\mathfrak{R}$-module with basis $(e_\alpha)_{\alpha\in\Phi^+}$ and define the linear maps $\varphi_i \in \mathscr{L}(V)$, $i \in I$, as in definition 7.

3.1. **Characterization of LK-families.**

The following proposition gives a characterization of an LK-family $(f_i)_{i\in I}$ in terms of relations between the $\mathfrak{f}_{i,\alpha} = f_i(e_\alpha)$'s. This generalizes [8, Prop. 3.2] and the computations of [12, proof of Thm. 3.8] and [21, proofs of lemmas 3.6 and 3.7].

Proposition 22. *An element $(f_i)_{i\in I} \in (V^\star)^I$ is an LK-family if and only if the relations listed in* Table 1 *below hold among the elements $\mathfrak{f}_{i,\alpha}$, $(i,\alpha) \in I \times \Phi^+$. (Note that the relations of cases* (6) *and* (8) *must hold whether $(\alpha_i|\alpha)$ is positive, zero or negative.)*

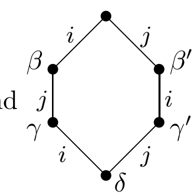
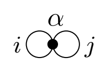
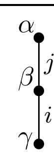
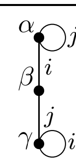
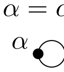
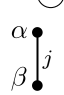
n°	Relations among the $\mathfrak{f}_{i,\alpha}$'s	Configuration of roots
(1)	$\mathfrak{f}_{i,\alpha_j} = 0$	if $i \neq j$
(2)	$\mathfrak{f}_{i,\alpha_i} = \mathfrak{f}_{j,\alpha_j}$	if $m_{i,j} = 3$
(3) (4)	$\mathfrak{f}_{i,\beta'} = \mathfrak{f}_{j,\beta}$ $\mathfrak{c}\mathfrak{f}_{i,\gamma'} + \mathfrak{a}\mathfrak{f}_{i,\delta} = \mathfrak{c}\mathfrak{f}_{j,\gamma} + \mathfrak{a}\mathfrak{f}_{j,\delta}$	if $m_{i,j} = 3$ and 
(5)	$\mathfrak{f}_{i,\alpha} = \mathfrak{f}_{j,\alpha}$	if $m_{i,j} = 3$ and 
(6)	$\mathfrak{d}\mathfrak{f}_{i,\alpha} = \mathfrak{b}\mathfrak{f}_{i,\beta}$	if $m_{i,j} = 2$ and 
(7)	$\mathfrak{c}\mathfrak{f}_{i,\alpha_i+\alpha_j} = -\mathfrak{a}\mathfrak{f}_{i,\alpha_i}$	if $m_{i,j} = 3$
(8)	$\mathfrak{c}\mathfrak{f}_{i,\alpha} = \mathfrak{b}\mathfrak{f}_{j,\gamma} - \mathfrak{a}\mathfrak{f}_{i,\beta}$	if $m_{i,j} = 3$ and 
(9) (10)	$\mathfrak{c}\mathfrak{f}_{i,\beta} = \mathfrak{d}\mathfrak{f}_{j,\gamma} - \mathfrak{a}\mathfrak{f}_{i,\gamma}$ $\mathfrak{d}\mathfrak{f}_{i,\alpha} = \mathfrak{b}\mathfrak{f}_{j,\beta}$	if $m_{i,j} = 3$ and 

Table 1. Relations for an LK-family.

Proof. This is simply the transcription on the basis elements e_α, $\alpha \in \Phi^+$, of V, of conditions (i), (ii) and (iii) of definition 12 on the linear forms f_i, $i \in I$. Indeed, relation (1) is condition (i), and for every i, $j \in I$, we get by definition :

$$\begin{cases} f_i\varphi_j(e_\alpha) = 0 & \text{if } \alpha = \alpha_j, \\ f_i\varphi_j(e_\alpha) = \mathfrak{d}\mathfrak{f}_{i,\alpha} & \text{if } \quad j, \\ \begin{cases} f_i\varphi_j(e_\alpha) = \mathfrak{b}\mathfrak{f}_{i,\beta} \\ f_i\varphi_j(e_\beta) = \mathfrak{a}\mathfrak{f}_{i,\beta} + \mathfrak{c}\mathfrak{f}_{i,\alpha} \end{cases} & \text{if } \quad \text{ in } \Phi^+. \end{cases}$$





Assume that $m_{i,j} = 2$. We thus have $f_i\varphi_j(e_\alpha) = \mathfrak{d} f_i(e_\alpha)$ if $\alpha = \alpha_j$ or if α is fixed by s_j. And for the $\{j\}$-mesh $\{\alpha, \beta\}$ displayed above,

$$\text{then } \begin{cases} f_i\varphi_j(e_\alpha) = \mathfrak{d} f_i(e_\alpha) \\ f_i\varphi_j(e_\beta) = \mathfrak{d} f_i(e_\beta) \end{cases} \iff \begin{cases} \mathfrak{b}\mathfrak{f}_{i,\beta} = \mathfrak{d}\mathfrak{f}_{i,\alpha} \\ \mathfrak{a}\mathfrak{f}_{i,\beta} + \mathfrak{c}\mathfrak{f}_{i,\alpha} = \mathfrak{d}\mathfrak{f}_{i,\beta} \end{cases},$$

and both equations give relation (6) since $\dfrac{\mathfrak{b}}{\mathfrak{d}} = \dfrac{\mathfrak{d} - \mathfrak{a}}{\mathfrak{c}}$ in $\mathfrak{R}$.

Now assume that $m_{i,j} = 3$, and consider the system of equations $f_i\varphi_j(e_\alpha) = f_j\varphi_i(e_\alpha)$, for α running through the vertices of a given $\{i, j\}$-mesh M, and for the four possible types of M (see remark 4).

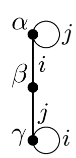
Type 5 : $M = \{\alpha_i, \alpha_j, \alpha_i + \alpha_j\}$ (the situations for α_i and α_j are symmetrical),

$$\text{then } \begin{cases} f_i\varphi_j(e_{\alpha_i}) = f_j\varphi_i(e_{\alpha_i}) \\ f_i\varphi_j(e_{\alpha_i+\alpha_j}) = f_j\varphi_i(e_{\alpha_i+\alpha_j}) \end{cases} \iff \begin{cases} \mathfrak{a}\mathfrak{f}_{i,\alpha_i} + \mathfrak{c}\mathfrak{f}_{i,\alpha_i+\alpha_j} = 0 \\ \mathfrak{b}\mathfrak{f}_{i,\alpha_i} = \mathfrak{b}\mathfrak{f}_{j,\alpha_j} \end{cases},$$

this gives relations (7) and (2) (since $\mathfrak{b} \in \mathfrak{R}^\times$).

Type 6 : $M = \{\alpha\}$, $f_i\varphi_j(e_\alpha) = f_j\varphi_i(e_\alpha) \iff \mathfrak{d}\mathfrak{f}_{i,\alpha} = \mathfrak{d}\mathfrak{f}_{j,\alpha}$, this is relation (5).

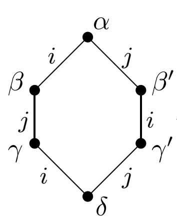
Type 7 : $M = \{\alpha, \beta, \gamma\}$ as displayed above,



$$\text{then } \begin{cases} f_i\varphi_j(e_\alpha) = f_j\varphi_i(e_\alpha) \\ f_i\varphi_j(e_\beta) = f_j\varphi_i(e_\beta) \\ f_i\varphi_j(e_\gamma) = f_j\varphi_i(e_\gamma) \end{cases} \iff \begin{cases} \mathfrak{d}\mathfrak{f}_{i,\alpha} = \mathfrak{b}\mathfrak{f}_{j,\beta} \\ \mathfrak{b}\mathfrak{f}_{i,\gamma} = \mathfrak{a}\mathfrak{f}_{j,\beta} + \mathfrak{c}\mathfrak{f}_{j,\alpha} \\ \mathfrak{a}\mathfrak{f}_{i,\gamma} + \mathfrak{c}\mathfrak{f}_{i,\beta} = \mathfrak{d}\mathfrak{f}_{j,\gamma} \end{cases},$$

this gives relations (10), one case of (8) (by exchanging i and j) and (9).

Type 8 : $M = \{\alpha, \beta, \beta', \gamma, \gamma', \delta\}$ as displayed above (the situations for β and β', and for γ and γ', are symmetrical),



$$\text{then } \begin{cases} f_i\varphi_j(e_\alpha) = f_j\varphi_i(e_\alpha) \\ f_i\varphi_j(e_\beta) = f_j\varphi_i(e_\beta) \\ f_i\varphi_j(e_\gamma) = f_j\varphi_i(e_\gamma) \\ f_i\varphi_j(e_\delta) = f_j\varphi_i(e_\delta) \end{cases} \iff \begin{cases} \mathfrak{b}\mathfrak{f}_{i,\beta'} = \mathfrak{b}\mathfrak{f}_{j,\beta} \\ \mathfrak{b}\mathfrak{f}_{i,\gamma} = \mathfrak{a}\mathfrak{f}_{j,\beta} + \mathfrak{c}\mathfrak{f}_{j,\alpha} \\ \mathfrak{a}\mathfrak{f}_{i,\gamma} + \mathfrak{c}\mathfrak{f}_{i,\beta} = \mathfrak{b}\mathfrak{f}_{j,\delta} \\ \mathfrak{a}\mathfrak{f}_{i,\delta} + \mathfrak{c}\mathfrak{f}_{i,\gamma'} = \mathfrak{a}\mathfrak{f}_{j,\delta} + \mathfrak{c}\mathfrak{f}_{j,\gamma} \end{cases},$$

this gives relations (3) (since $\mathfrak{b} \in \mathfrak{R}^\times$), the two last cases of (8), and (4). □

Note that these relations are of two kinds : relations (1) to (5) give equalities between elements associated with roots of the same depth, whereas relations (6) to (10) express an element $\mathfrak{f}_{i,\alpha}$ in terms of a linear combination of some $\mathfrak{f}_{j,\beta}$'s with $\mathrm{dp}(\beta) < \mathrm{dp}(\alpha)$. In fact, relation (5) can be deleted from Table 1 (this generalizes the analogous observation of [8, proof of Prop. 3.2], [12, proof of Thm. 3.8], and [21, Lem. 3.5]), this is proved in lemma 24 below.

Lemma 23. *Let i, j, $k \in I$ be such that $m_{i,j} = m_{j,k} = m_{k,i} = 3$. Then for every $\alpha \in \Phi^+$, we have $(\alpha_i|\alpha) + (\alpha_j|\alpha) + (\alpha_k|\alpha) \leqslant 0$.*

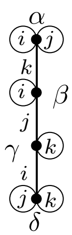
Proof. The map $v \mapsto (\alpha_i|v) + (\alpha_j|v) + (\alpha_k|v)$ is a linear form on $E = \oplus_{l\in I}\mathbb{R}\alpha_l$ which is clearly non-positive on the basis elements α_l, $l \in I$. This gives the result since every $\alpha \in \Phi^+$ is a linear combination, with non-negative coefficients, of those elements α_l, $l \in I$. □

Lemma 24. *Relation* (5) *of* Table 1 *is implied by relations* (1), (6), (8) *and* (10).

Proof. Fix $i,\, j \in I$ with $m_{i,j} = 3$ and $\alpha \in \Phi^+$ such that $(\alpha_i|\alpha) = (\alpha_j|\alpha) = 0$. Let us show by induction on $\mathrm{dp}(\alpha)$ that $\mathfrak{f}_{i,\alpha} = \mathfrak{f}_{j,\alpha}$, using only relations (1), (6), (8) and (10). If $\mathrm{dp}(\alpha) = 1$, *i.e.* if $\alpha = \alpha_k$ for some $k \in I$, then $k \neq i,\, j$ and the result is given by (1). So assume that $\mathrm{dp}(\alpha) \geqslant 2$ and fix $k \in I$ such that $(\alpha_k|\alpha) > 0$; we set $\beta = s_k(\alpha) \in \Phi^+$.

If $m_{i,k} = m_{j,k} = 2$, then we get $(\alpha_i|\beta) = (\alpha_j|\beta) = 0$, whence $\mathfrak{f}_{i,\beta} = \mathfrak{f}_{j,\beta}$ by induction and hence $\mathfrak{f}_{i,\alpha} = \mathfrak{f}_{j,\alpha}$ by (6) since $\mathfrak{d} \in \mathfrak{R}^\times$.

If $m_{i,k} = 2$ and $m_{j,k} = 3$, then the graph of $[\alpha]_{\{i,j,k\}}$ is the following :



, whence
$$\left|\begin{array}{lll} \mathfrak{f}_{i,\alpha} & = \dfrac{\mathfrak{b}}{\mathfrak{d}}\mathfrak{f}_{i,\beta} & \text{by (6)} \\ & = \dfrac{\mathfrak{b}}{\mathfrak{c}\mathfrak{d}}\left(\mathfrak{b}\mathfrak{f}_{j,\delta} - \mathfrak{a}\mathfrak{f}_{i,\gamma}\right) & \text{by (8)} \\ \mathfrak{f}_{j,\alpha} & = \dfrac{1}{\mathfrak{c}}\left(\mathfrak{b}\mathfrak{f}_{k,\gamma} - \mathfrak{a}\mathfrak{f}_{j,\beta}\right) & \text{by (8)} \\ & = \dfrac{\mathfrak{b}}{\mathfrak{c}\mathfrak{d}}\left(\mathfrak{b}\mathfrak{f}_{k,\delta} - \mathfrak{a}\mathfrak{f}_{i,\gamma}\right) & \text{by (6) and (10)} \\ \mathfrak{f}_{j,\delta} & = \mathfrak{f}_{k,\delta} & \text{by induction} \end{array}\right.$$
, and hence $\mathfrak{f}_{i,\alpha} = \mathfrak{f}_{j,\alpha}$.

Thanks to lemma 23, we cannot have $m_{i,k} = m_{j,k} = 3$ in that situation, so we are done (up to exchanging i and j). □

Remark 25. LK-families and reducibility.

If $\Gamma_1, \ldots, \Gamma_p$ are the connected components of Γ, with vertex set $I_1, \ldots, I_p$ respectively, then $\Phi = \Phi_{I_1} \sqcup \cdots \sqcup \Phi_{I_p}$ and relations (1) and (6) imply that $\mathfrak{f}_{i,\alpha} = 0$ for every $(i, \alpha) \in I_m \times \Phi^+_{I_n}$ whenever $m \neq n$. As a consequence, any LK-representation ψ of B^+_Γ is the direct sum of the induced LK-representations ψ_n of $B^+_{\Gamma_n}$ for $1 \leqslant n \leqslant p$.

Hence when considering LK-representations (or LK-families), there is no loss of generality in assuming that Γ is connected, in which case the elements $\mathfrak{f}_{i,\alpha_i}$, $i \in I$, are all equal by relation (2).

3.2. **The spherical case.**

We assume here that $\Gamma = A_n$ $(n \geqslant 1)$, D_n $(n \geqslant 4)$, or E_n $(n = 6, 7 \text{ or } 8)$. In the following theorem, we rephrase the unicity statements of [8, Prop. 3.5] and [12, Thm. 3.8]. Recall that $\mathscr{F} \subseteq (V^\star)^I$ is the $\mathfrak{R}$-module of LK-families.

Definition 26. Fix $i_0 \in I$. Let μ be the linear map $\mathscr{F} \to \mathfrak{R},\ (f_i)_{i\in I} \mapsto \mathfrak{f}_{i_0,\alpha_{i_0}}$. In view of relation (2) (and of the fact that Γ is connected), μ does not depend on the choice of $i_0 \in I$.

Theorem 27. *The linear map $\mu : \mathscr{F} \to \mathfrak{R}$, is an isomorphism (or $\mathfrak{R}$-modules).*

Proof. Since Γ is spherical, there is no mesh of type 8 (see remark 4) in Φ^+.

Hence for a given LK-family $(f_i)_{i\in I}$, every $\mathfrak{f}_{i,\alpha}$ with $\mathrm{dp}(\alpha) \geqslant 2$ can be expressed as a linear combination of some $\mathfrak{f}_{j,\beta}$'s with $\mathrm{dp}(\beta) < \mathrm{dp}(\alpha)$, via at least one of the relations (6) to (10). As a consequence, $(f_i)_{i\in I}$ is entirely determined by the values of the $\mathfrak{f}_{i,\alpha_j}$, for $i,\, j \in I$. And since $\mathfrak{f}_{i,\alpha_j} = 0$ if $i \neq j$ by (1), and $\mathfrak{f}_{i,\alpha_i} = \mathfrak{f}_{i_0,\alpha_{i_0}}$ for every $i \in I$ by (2) (since Γ is connected), $(f_i)_{i\in I}$ is in fact entirely determined by the value $\mathfrak{f}_{i_0,\alpha_{i_0}}$, whence the injectivity of μ.

In order to show its surjectivity, the idea is to define an LK-family inductively, with basis step $\mathfrak{f}_{i,\alpha_i} = \mathfrak{f} \in \mathfrak{R}$ (one could chose $\mathfrak{f} = 1$ by linearity) and $\mathfrak{f}_{i,\alpha_j} = 0$ if $i \neq j$, and inductive step one of the suitable relations (6) to (10) to define $\mathfrak{f}_{i,\alpha}$ (with

$\mathrm{dp}(\alpha) \geqslant 2$) in terms of a linear combination of some $\mathfrak{f}_{j,\beta}$'s with $\mathrm{dp}(\beta) < \mathrm{dp}(\alpha)$. Proving that the obtained family is indeed an LK-family amounts to proving that the definition of $\mathfrak{f}_{i,\alpha}$ does not depend on the choice of the suitable relation chosen in the inductive step. This is essentially done in [8, Prop. 3.5] and [12, Thm. 3.8]. $\square$

When Γ is connected and spherical (and of small type), LK-representations of B^+ are then parametrized by $\mathfrak{R}$ and LK-representations of B (those corresponding to LK-families with $\mathfrak{f}_{i_0,\alpha_{i_0}} \in \mathfrak{R}^\times$) are parametrized by $\mathfrak{R}^\times$.

Let us end this subsection with some consequences of that construction.

Note that since Γ is spherical, the free $\mathfrak{R}$-module V is finite-dimensional and hence the notion of *determinant* of an element of $\mathscr{L}(V)$ is defined.

Corollary 28. *Tow LK-representations ψ and ψ', associated with to distinct LK-families $(f_i)_{i\in I}$ and $(f'_i)_{i\in I}$ respectively, are non-equivalent.*

Proof. It suffices to see that for a given $i \in I$, the maps ψ_i and ψ'_i have distinct determinant. But in view of remark 8, we get $\det(\psi_i) = \mathfrak{u}\mathfrak{f}_{i,\alpha_i}$ and $\det(\psi'_i) = \mathfrak{u}\mathfrak{f}'_{i,\alpha_i}$ for a certain $\mathfrak{u} \in \mathfrak{R}^\times$, whence the result since the previous theorem shows that $(f_i)_{i\in I} \neq (f'_i)_{i\in I}$ implies $\mathfrak{f}_{i,\alpha_i} \neq \mathfrak{f}'_{i,\alpha_i}$. $\square$

Finally, an easy induction on $\mathrm{dp}(\alpha)$ gives the following remark, which generalizes [8, Cor. 3.3] :

Remark 29. Let $(f_i)_{i\in I}$ be an LK-family and set $f_{i_0,\alpha_{i_0}} = \mathfrak{f}$.

Then for every $\alpha \neq \alpha_i$, we have $\mathfrak{f}_{i,\alpha} \in -\dfrac{\mathfrak{a}\mathfrak{f}}{\mathfrak{c}}\mathfrak{R}$, and more precisely :

(i) $\mathfrak{f}_{i,\alpha} = 0$ if $i \notin \mathrm{Supp}(\alpha)$, and

(ii) $\mathfrak{f}_{i,\alpha} = -\dfrac{\mathfrak{a}\mathfrak{f}}{\mathfrak{c}}\left(\dfrac{\mathfrak{b}}{\mathfrak{d}}\right)^{\mathrm{dp}(\alpha)-2}$ if $\mathrm{dp}(\alpha) \geqslant 2$ and $(\alpha_i|\alpha) > 0$.

This construction can be generalized to an arbitrary Coxeter matrix of small type with no triangle, following [21]. This is done in the following subsection.

3.3. **The LK-family of Paris.**

The main construction of [21] is a uniform construction of an LK-family with $\mathfrak{f}_{i,\alpha_i} \in \mathfrak{R}^\times$ for every $i \in I$, for any Coxeter matrix of small type $\Gamma = (m_{i,j})_{i,j\in I}$ with no triangle, *i.e.* no subset $\{i,j,k\} \subseteq I$ with $m_{i,j} = m_{j,k} = m_{k,i} = 3$.

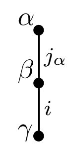
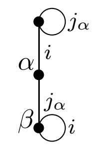
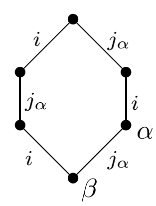
This construction is made over $\mathfrak{R} = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$, with $\mathfrak{f}_{i,\alpha_i} = xy^2$ for every $i \in I$. The aim of this subsection is to generalize it to our settings.

Definition 30 ([21])**.** For every $\alpha \in \Phi^+$ with $\mathrm{dp}(\alpha) \geqslant 2$, fix an element $j_\alpha \in I$ such that $(\alpha|\alpha_{j_\alpha}) > 0$. Let us define a family $(\mathfrak{f}_{i,\alpha})_{(i,\alpha)\in I\times\Phi^+}$ by induction on $\mathrm{dp}(\alpha)$ as follows :

- Basis step : fix $\mathfrak{f} \in \mathfrak{R}$ (one can choose $\mathfrak{f} = 1$ by linearity) and set

Case	Value of $\mathfrak{f}_{i,\alpha}$	Condition
(C1)	$\mathfrak{f}$	if $\alpha = \alpha_i$
(C2)	0	if $\alpha = \alpha_j$ for $j \neq i$
(C3)	$-\dfrac{\mathfrak{a}\mathfrak{f}}{\mathfrak{c}}\left(\dfrac{\mathfrak{b}}{\mathfrak{d}}\right)^{\mathrm{dp}(\alpha)-2}$	if $\mathrm{dp}(\alpha) \geqslant 2$ and $(\alpha|\alpha_i) > 0$

• Inductive step : if $\mathrm{dp}(\alpha) \geqslant 2$ and $(\alpha|\alpha_i) \leqslant 0$ — hence $i \neq j_\alpha$ — then set

Case	Value of $\mathfrak{f}_{i,\alpha}$	Condition
(C4)	$\frac{\mathfrak{b}}{\mathfrak{d}}\mathfrak{f}_{i,\beta}$	if $m_{i,j_\alpha} = 2$
(C5)	$\frac{1}{\mathfrak{c}}\left(\mathfrak{b}\mathfrak{f}_{j_\alpha,\gamma} - \mathfrak{a}\mathfrak{f}_{i,\beta}\right)$	if $m_{i,j_\alpha} = 3$ and 
(C6)	$\frac{1}{\mathfrak{c}}\left(\mathfrak{d}\mathfrak{f}_{j_\alpha,\beta} - \mathfrak{a}\mathfrak{f}_{i,\beta}\right)$	if $m_{i,j_\alpha} = 3$ and 
(C7)	$\frac{\mathfrak{b}}{\mathfrak{d}}\mathfrak{f}_{i,\beta} + \frac{\mathfrak{d}}{\mathfrak{c}}\mathfrak{f}_{j_\alpha,\beta} + \frac{\mathfrak{a}\mathfrak{d}\mathfrak{f}}{\mathfrak{c}^2}\left(\frac{\mathfrak{b}}{\mathfrak{d}}\right)^{\mathrm{dp}(\alpha)-3}$	if $m_{i,j} = 3$ and 

And we define the family $(f_i)_{i\in I} \in (V^\star)^I$ via $f_i(e_\alpha) = \mathfrak{f}_{i,\alpha}$ for $(i,\alpha) \in I \times \Phi^+$.

Note that cases (C4) and (C5) occur whether $(\alpha_i|\alpha)$ is zero or negative. Case (C3) is a generalization of what happens when Γ is spherical, but is no longer a consequence of the relations of Table 1 in general, and neither is case (C7) (see subsection 3.4).

Proposition 31. *Assume that the family $(f_i)_{i\in I}$ of definition 30 does not depend on the choice of the j_α's. Then it is an LK-family.*

Proof. We have to show that the family $(\mathfrak{f}_{i,\alpha})_{(i,\alpha)\in I\times\Phi^+}$ satisfies the relations (1) to (4) and (6) to (10) of proposition 22 (thanks to lemma 24). Relations (1), (2), (3), (7) and (10) are clearly satisfied by construction. In the same way, (C3) implies relations (6) and (8) when $(\alpha_i|\alpha) > 0$ (use $\mathfrak{d}(\mathfrak{a}-\mathfrak{d}) + \mathfrak{b}\mathfrak{c} = 0$ to establish (8)).

Now consider a relation (4), (6) with $(\alpha_i|\alpha) \leqslant 0$, (8) with $(\alpha_i|\alpha) \leqslant 0$, or (9). Then the elements $\mathfrak{f}_{i,\alpha}$, for α of highest depth among the roots involved in this relation, are defined by induction. Under the assumption of the proposition, we are free to use the suitable case among (C7), (C4), (C5) or (C6) respectively, to define them ; this clearly shows that the considered relation is satisfied (use the fact that $\mathfrak{d}(\mathfrak{a}-\mathfrak{d}) + \mathfrak{b}\mathfrak{c} = 0$ to establish relation (4) via (C7)). $\square$

The previous proposition generalizes the computations of [21, lemmas 3.5, 3.6 and 3.7]. It is not clear whether the independance assumption is true in general, but this is at least the case when Γ has no triangle :

Proposition 32. *Assume that Γ has no triangle. Then the family $(f_i)_{i\in I}$ of definition 30 does not depend on the choice of the j_α's.*

Proof. This is [21, lemmas 3.3 and 3.4] : our settings are slightly more general, but the (long) computations of the proofs are essentially the same. □

Hence if Γ has no triangle, then the module of LK-families $\mathscr{F}$ is not trivial (contains a free submodule of dimension 1), and, by choosing $\mathfrak{f} \in \mathfrak{R}^\times$ in definition 30 above, one obtains an element of $\mathscr{F}_{\mathrm{gr}}$.

It can also be shown that the family of definition 30 does not depend on the choice of the j_α's when $\Gamma = \tilde{A}_2$ (following the same steps as in the proof of lemma 37 below), hence the same holds for this triangle graph. We will more generally explicit all the LK-families for any affine Coxeter graph in the following section.

3.4. **The affine case.**

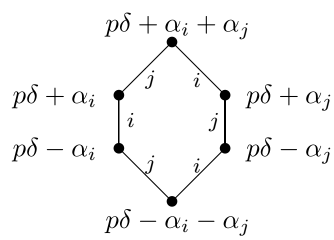
We assume here that $\Gamma = (m_{i,j})_{0 \leqslant i,j \leqslant n}$ is a Coxeter matrix of type $\tilde{A}_n$ ($n \geqslant 2$), $\tilde{D}_n$ ($n \geqslant 4$) or $\tilde{E}_n$ ($n = 6$, 7, 8). We set $I = [\![0, n]\!]$.

We denote by $\Gamma_0 = (m_{i,j})_{1 \leqslant i,j \leqslant n}$ the corresponding spherical Coxeter matrix. Let Φ (resp. Φ_0) be the root system associated with Γ (resp. Γ_0) in $E = \oplus_{i=0}^n \mathbb{R}\alpha_i$ (resp. $E_0 = \oplus_{i=1}^n \mathbb{R}\alpha_i$) and let δ be the first positive imaginary root of Φ, then we have the following decomposition (see [15]) :

$$\Phi = \bigsqcup_{p \in \mathbb{Z}} \big(\Phi_0 + p\delta\big) \quad \text{and} \quad \Phi^+ = \Phi_0^+ \bigsqcup \Big(\bigsqcup_{p \in \mathbb{N}_{\geqslant 1}} \big(\Phi_0 + p\delta\big)\Big).$$

As a consequence, we get the following remark :

Remark 33. The only meshes of type 8 in Φ^+ (see remark 4) are the following ones, for $p \geqslant 1$ and $m_{i,j} = 3$:



In particular, for a given $(i, \alpha) \in I \times \Phi^+$ with $\mathrm{dp}(\alpha) \geqslant 2$, then either $\alpha = p\delta \pm \alpha_i$ for some $p \geqslant 1$, or the pair (i, α) appears at the left-hand side of (at least) one of the relations (6) to (10) of Table 1, and for every such relation and every pair (j, β) involved in its right-hand side, then $\beta \neq q\delta - \alpha_j$ for every $q \geqslant 1$.

Definition 34. Let i_0, $j_0 \in I$ be such that $m_{i_0,j_0} = 3$. We denote by μ the linear map $\mathscr{F} \to \mathfrak{R}^{\mathbb{N}}$, $(f_i)_{i \in I} \mapsto (\mathfrak{f}_n)_{n \in \mathbb{N}}$, where

(i) $\mathfrak{f}_{2p} = \mathfrak{f}_{i_0, p\delta + \alpha_{i_0}}$ for every $p \in \mathbb{N}$,
(ii) $\mathfrak{f}_{2p-1} = \mathfrak{c}\mathfrak{d}\mathfrak{f}_{i_0, p\delta - \alpha_{i_0}} - \mathfrak{b}\mathfrak{c}\mathfrak{f}_{i_0, p\delta - \alpha_{i_0} - \alpha_{j_0}} - \mathfrak{d}^2 \mathfrak{f}_{j_0, p\delta - \alpha_{i_0} - \alpha_{j_0}}$ for every $p \in \mathbb{N}_{\geqslant 1}$.

We will show in proposition 39 below that μ does not depend on the choice of i_0, $j_0 \in I$ such that $m_{i_0,j_0} = 3$. The aim of this subsection is then to prove the following :

Theorem 35. *The linear map $\mu : \mathscr{F} \to \mathfrak{R}^{\mathbb{N}}$ is an isomorphism (of $\mathfrak{R}$-modules).*

Hence in those cases, LK-representations of B^+ are parametrized by $\mathfrak{R}^{\mathbb{N}}$ and LK-representations of B (those corresponding to LK-families with $\mathfrak{f}_{i_0,\alpha_{i_0}} \in \mathfrak{R}^\times$) are parametrized by $\mathfrak{R}^\times \times \mathfrak{R}^{\mathbb{N}_{\geqslant 1}}$. The injectivity and surjectivity of μ are proved respectively in propositions 40 and 41 below.

Notation 36. For every $k \in \mathbb{N}_{\geqslant 1}$, let us denote by Φ_k^+ the subset of Φ^+ composed of the positive roots of depth smaller than (or equal to) k.

In the two following lemmas, we assume that we are given a family $\mathfrak{F}_k = (\mathfrak{f}_{i,\alpha})_{(i,\alpha)\in I\times\Phi_k^+} \in \mathfrak{R}^{I\times\Phi_k^+}$ whose elements satisfy the relations of Table 1 whenever the roots involved are of depth smaller than (or equal to) k, and it is understood that we work with the elements of $\mathfrak{F}_k$.

Lemma 37. *Fix $(i,p) \in I \times \mathbb{N}_{\geqslant 1}$ and assume that $k = \mathrm{dp}(p\delta - \alpha_i) - 1$. Then the element $\mathfrak{bc}\mathfrak{f}_{i,p\delta-\alpha_i-\alpha_j} + \mathfrak{d}^2\mathfrak{f}_{j,p\delta-\alpha_i-\alpha_j}$ does not depend on $j \in I$ such that $m_{i,j} = 3$.*

Proof. Assume that $j,\, k \in I$ are such that $m_{i,j} = m_{i,k} = 3$.

If $m_{j,k} = 2$, then the result follows from relations (6) and (9) : indeed, we get

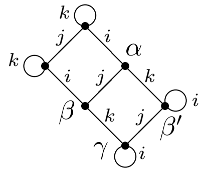
$$\begin{array}{rcll}
\mathfrak{c}\mathfrak{f}_{i,p\delta-\alpha_i-\alpha_j} & = & \mathfrak{d}\mathfrak{f}_{k,p\delta-\alpha_i-\alpha_j-\alpha_k} - \mathfrak{a}\mathfrak{f}_{i,p\delta-\alpha_i-\alpha_j-\alpha_k} & \text{by (9)},\\
\mathfrak{c}\mathfrak{f}_{i,p\delta-\alpha_i-\alpha_k} & = & \mathfrak{d}\mathfrak{f}_{j,p\delta-\alpha_i-\alpha_j-\alpha_k} - \mathfrak{a}\mathfrak{f}_{i,p\delta-\alpha_i-\alpha_j-\alpha_k} & \text{by (9)},\\
\mathfrak{d}\mathfrak{f}_{j,p\delta-\alpha_i-\alpha_j} & = & \mathfrak{b}\mathfrak{f}_{j,p\delta-\alpha_i-\alpha_j-\alpha_k} & \text{by (6)},\\
\mathfrak{d}\mathfrak{f}_{k,p\delta-\alpha_i-\alpha_k} & = & \mathfrak{b}\mathfrak{f}_{k,p\delta-\alpha_i-\alpha_j-\alpha_k} & \text{by (6)}.
\end{array}$$

If $m_{j,k} = 3$, then $\Gamma = \tilde{A}_2$, $\{i,j,k\} = \{0,1,2\}$ and $p\delta - \alpha_i - \alpha_j = (p-1)\delta + \alpha_k$. In that case we can prove more, namely that the value of $\mathfrak{f}_{l,(p-1)\delta+\alpha_m}$ does not depend on the pair $(l,m) \in \{0,1,2\}^2$ such that $l \neq m$. To do this, one can first prove the similar statement for the elements $\mathfrak{f}_{l,q\delta-\alpha_m}$ with $1 \leqslant q \leqslant p-1$ by induction on q, thanks to relations (3) and (8) (the case $q = 1$ is given by relations (7) and (2)), and then prove the desired statement for the elements $\mathfrak{f}_{l,q\delta+\alpha_m}$ with $0 \leqslant q \leqslant p-1$ by induction on q, thanks to relation (8) and the intermediate result (the case $q = 0$ is given by relation (1)). □

Lemma 38. *Fix $(i,\alpha) \in I \times \Phi^+$, with $\alpha \neq p\delta \pm \alpha_i$ for every $p \in \mathbb{N}$, and assume that $k = \mathrm{dp}(\alpha) - 1$. If we define $\mathfrak{f}_{i,\alpha} \in \mathfrak{R}$ by one of the relations (6) to (10) where (i,α) appears at the left-hand side, then the value of $\mathfrak{f}_{i,\alpha}$ does not depend on the chosen relation.*

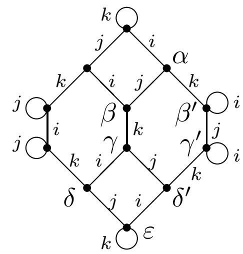
Proof. Let us assume that (i,α) appears at the left-hand side of two of the relations (6) to (10) and let us denote by j (resp. k) the index distinct from i involved in the first (resp. the second) of those two relations. Note that, in that situation, then $\Gamma \neq \tilde{A}_2$ and there are only ten possibilities up to exchanging j and k (use remark 33) : two relations (6) with $m_{j,k} = 2$ or $m_{j,k} = 3$, relations (6) and (8) with $m_{j,k} = 2$, relations (6) and (9) with $m_{j,k} = 2$ or $m_{j,k} = 3$, relations (6) and (10) with $m_{j,k} = 2$ or $m_{j,k} = 3$, two relations (8) with $m_{j,k} = 2$, two relations (9) with $m_{j,k} = 2$, two relations (10) with $m_{j,k} = 2$.

For example in the case (6) and (9) with $m_{j,k} = 2$, the graph of $[\alpha]_{\{i,j,k\}}$ is



$$\text{whence}\left\{\begin{array}{rcll} \dfrac{\mathfrak{b}}{\mathfrak{d}}\mathfrak{f}_{i,\beta} &=& \dfrac{\mathfrak{b}}{\mathfrak{c}}\mathfrak{f}_{k,\gamma}-\dfrac{\mathfrak{a}\mathfrak{b}}{\mathfrak{c}\mathfrak{d}}\mathfrak{f}_{i,\gamma} & \text{by (9), and}\\ \dfrac{\mathfrak{d}}{\mathfrak{c}}\mathfrak{f}_{k,\beta'}-\dfrac{\mathfrak{a}}{\mathfrak{c}}\mathfrak{f}_{i,\beta'} &=& \dfrac{\mathfrak{b}}{\mathfrak{c}}\mathfrak{f}_{k,\gamma}-\dfrac{\mathfrak{a}\mathfrak{b}}{\mathfrak{c}\mathfrak{d}}\mathfrak{f}_{i,\gamma} & \text{by (6) (two times).} \end{array}\right.$$

And in the case (6) and (9) with $m_{j,k}=3$, the graph of $[\alpha]_{\{i,j,k\}}$ is



$$\text{whence}\left\{\begin{array}{rcll} \dfrac{\mathfrak{b}}{\mathfrak{d}}\mathfrak{f}_{i,\beta} &=& \dfrac{\mathfrak{b}^2}{\mathfrak{c}\mathfrak{d}}\mathfrak{f}_{k,\delta}-\dfrac{\mathfrak{a}\mathfrak{b}}{\mathfrak{c}\mathfrak{d}}\mathfrak{f}_{i,\gamma} & \text{by (8),}\\ &=& \dfrac{\mathfrak{b}^2}{\mathfrak{c}^2}\mathfrak{f}_{j,\varepsilon}-\dfrac{\mathfrak{a}\mathfrak{b}^2}{\mathfrak{c}^2\mathfrak{d}}\mathfrak{f}_{k,\varepsilon}-\dfrac{\mathfrak{a}\mathfrak{b}^2}{\mathfrak{c}\mathfrak{d}^2}\mathfrak{f}_{i,\delta'} & \text{by (9) and (6),}\\ \dfrac{\mathfrak{d}}{\mathfrak{c}}\mathfrak{f}_{k,\beta'}-\dfrac{\mathfrak{a}}{\mathfrak{c}}\mathfrak{f}_{i,\beta'} && & \\ &=& \dfrac{\mathfrak{b}\mathfrak{d}}{\mathfrak{c}^2}\mathfrak{f}_{j,\delta'}-\dfrac{\mathfrak{a}\mathfrak{d}}{\mathfrak{c}^2}\mathfrak{f}_{k,\gamma'}-\dfrac{\mathfrak{a}\mathfrak{b}}{\mathfrak{c}\mathfrak{d}}\mathfrak{f}_{i,\gamma'} & \text{by (8) and (6),}\\ &=& \dfrac{\mathfrak{b}^2}{\mathfrak{c}^2}\mathfrak{f}_{j,\varepsilon}-\dfrac{\mathfrak{a}\mathfrak{b}}{\mathfrak{c}^2}\mathfrak{f}_{i,\delta'}-\dfrac{\mathfrak{a}\mathfrak{b}^2}{\mathfrak{c}^2\mathfrak{d}}\mathfrak{f}_{k,\varepsilon}+\dfrac{\mathfrak{a}^2\mathfrak{b}}{\mathfrak{c}^2\mathfrak{d}}\mathfrak{f}_{i,\delta'} & \text{by (6), (10) and (8),}\\ &=& \dfrac{\mathfrak{b}^2}{\mathfrak{c}^2}\mathfrak{f}_{j,\varepsilon}-\dfrac{\mathfrak{a}\mathfrak{b}^2}{\mathfrak{c}^2\mathfrak{d}}\mathfrak{f}_{k,\varepsilon}+\dfrac{\mathfrak{a}\mathfrak{b}(\mathfrak{a}-\mathfrak{d})}{\mathfrak{c}^2\mathfrak{d}}\mathfrak{f}_{i,\delta'}, & \end{array}\right.$$

and the result since $\frac{\mathfrak{a}\mathfrak{b}(\mathfrak{a}-\mathfrak{d})}{\mathfrak{c}^2\mathfrak{d}}+\frac{\mathfrak{a}\mathfrak{b}^2}{\mathfrak{c}\mathfrak{d}^2}=\frac{\mathfrak{a}\mathfrak{b}}{(\mathfrak{c}\mathfrak{d})^2}(\mathfrak{d}(\mathfrak{a}-\mathfrak{d})+\mathfrak{b}\mathfrak{c})=0$.

The eight remaining cases are similar and left to the reader. □

We are now able to prove the announced results on the linear map $\mu:\mathscr{F}\to\mathfrak{R}^{\mathbb{N}}$ of definition 34.

Proposition 39. *The definition of μ does not depend on the choice of i_0, $j_0\in I$ such that $m_{i_0,j_0}=3$.*

Proof. Let $(f_i)_{i\in I}$ be an LK-family. Relations (2) and the fact Γ is connected show that the elements $\mathfrak{f}_{i,\alpha_i}$, $i\in I$, are all equal to $\mathfrak{f}_0$. Now fix $p\in\mathbb{N}_{\geqslant 1}$. In the same vein, relation (3) and the fact Γ is connected show that the elements $\mathfrak{f}_{i,p\delta+\alpha_i}$, $i\in I$, are all equal to $\mathfrak{f}_{2p}$.

Now if we set $\mathfrak{f}_{i,j}=\mathfrak{c}\mathfrak{d}\mathfrak{f}_{i,p\delta-\alpha_i}-\mathfrak{b}\mathfrak{c}\mathfrak{f}_{i,p\delta-\alpha_i-\alpha_j}-\mathfrak{d}^2\mathfrak{f}_{j,p\delta-\alpha_i-\alpha_j}$ for i, $j\in I$ with $m_{i,j}=3$, we are left to show that $\mathfrak{f}_{i,j}=\mathfrak{f}_{i_0,j_0}=\mathfrak{f}_{2p-1}$ for every i, $j\in I$ with $m_{i,j}=3$. But lemma 37 gives $\mathfrak{f}_{i,j}=\mathfrak{f}_{i,k}$ if $m_{i,j}=m_{j,k}=3$, and since $\mathfrak{a}=\mathfrak{d}-\frac{\mathfrak{b}\mathfrak{c}}{\mathfrak{d}}$, relation (4) can be written $\frac{1}{\mathfrak{d}}\mathfrak{f}_{i,j}=\frac{1}{\mathfrak{d}}\mathfrak{f}_{j,i}$, whence $\mathfrak{f}_{i,j}=\mathfrak{f}_{j,i}$, when $m_{i,j}=3$. The connectedness of Γ then gives the result. □

Proposition 40. *The linear map μ is injective.*

Proof. Let $(f_i)_{i\in I}$ be an LK-family. Thanks to remark 33, an easy induction on $\mathrm{dp}(\alpha)$ shows that, for every $(i,\alpha)\in[\![0,n]\!]\times\Phi^+$, either $\alpha=p\delta-\alpha_i$ for some $p\in\mathbb{N}_{\geqslant 1}$, or $\mathfrak{f}_{i,\alpha}$ is a linear combination of some $\mathfrak{f}_{j,p\delta+\alpha_j}$'s for $j\in I$ and $p\in\mathbb{N}$. Hence $(f_i)_{i\in I}$ is entirely determined by the elements $\mathfrak{f}_{i,p\delta\pm\alpha_i}$, for $i\in I$ and $p\in\mathbb{N}$.

But by proposition 39, the elements $\mathfrak{f}_{i,p\delta+\alpha_i}$, $i\in I$, are all equal to $\mathfrak{f}_{2p}$ and we get $\mathfrak{f}_{i,p\delta-\alpha_i}=\frac{\mathfrak{b}}{\mathfrak{d}}\mathfrak{f}_{i,p\delta-\alpha_i-\alpha_j}+\frac{\mathfrak{d}}{\mathfrak{c}}\mathfrak{f}_{j,p\delta-\alpha_i-\alpha_j}+\frac{1}{\mathfrak{c}\mathfrak{d}}\mathfrak{f}_{2p-1}$ for any $j\in I$ with $m_{i,j}=3$, hence $\mathfrak{f}_{i,p\delta-\alpha_i}$ is entirely determined by some $\mathfrak{f}_{2q}$, $q\in\mathbb{N}$, and $\mathfrak{f}_{2p-1}$. □

Proposition 41. *The linear map μ is surjective.*

Proof. Fix $(\mathfrak{f}_n)_{n\in\mathbb{N}} \in \mathfrak{R}^{\mathbb{N}}$, and let us construct a family $(f_i)_{i\in I} \in (V^\star)^I$ by induction as follows (recall that we set $f_i(e_\alpha) = \mathfrak{f}_{i,\alpha}$ for every $(i,\alpha) \in I \times \Phi^+$) :

- Basis step : we set $\mathfrak{f}_{i,\alpha_j} = 0$ for $i \neq j$, and $\mathfrak{f}_{i,p\delta+\alpha_i} = \mathfrak{f}_{2p}$ for every $i \in I$ and $p \in \mathbb{N}$.
- Inductive step : if $(i,\alpha) \in I \times \Phi^+$ is not handled by the basis step (hence $\mathrm{dp}(\alpha) \geqslant 2$) and is such that all the $\mathfrak{f}_{j,\beta}$, for $j \in I$ and $\mathrm{dp}(\beta) < \mathrm{dp}(\alpha)$, are constructed. Then
  (i) if $\alpha = p\delta - \alpha_i$, we set $\mathfrak{f}_{i,\alpha} = \dfrac{\mathfrak{b}}{\mathfrak{d}}\mathfrak{f}_{i,p\delta-\alpha_i-\alpha_j} + \dfrac{\mathfrak{d}}{\mathfrak{c}}\mathfrak{f}_{j,p\delta-\alpha_i-\alpha_j} + \dfrac{1}{\mathfrak{c}\mathfrak{d}}\mathfrak{f}_{2p-1}$ for some j such that $m_{i,j} = 3$,
  (ii) if not, then (i,α) appears at the left-hand side of (at least) one of the relations (6) to (10) (see remark 33) ; we define $\mathfrak{f}_{i,\alpha}$ via the corresponding right-hand side.

We are left to show that $(f_i)_{i\in I} \in (V^\star)^I$ is an LK-family, since it will then be, by construction, an antecedent of $(\mathfrak{f}_n)_{n\in\mathbb{N}} \in \mathfrak{R}^{\mathbb{N}}$ by μ. We proceed by induction on $m \in \mathbb{N}$ in order to show that the relations of TABLE 1 that involve only roots of depth smaller than (or equal to) m are satisfied by the elements $\mathfrak{f}_{i,\alpha}$, for $i \in I$ and $\alpha \in \Phi^+$ with $\mathrm{dp}(\alpha) \leqslant m$.

If $m = 0$, the only relations to consider are relations (1) and (2), which are satisfied by construction of the basis step. Relation (3) is also satisfied for arbitrary depths by construction of the basis step. Now assume that we know the result for some $m \in \mathbb{N}$ and consider a relation that involves a root of depth $m+1$ and no root of higher depth.

Assume first that it is a relation of type (4), involving the indices i and j (and hence the roots $p\delta - \alpha_i$, $p\delta - \alpha_j$ and $p\delta - \alpha_i - \alpha_j$). Lemma 37 shows that the definition of $\mathfrak{f}_{i,p\delta-\alpha_i}$ (resp. $\mathfrak{f}_{j,p\delta-\alpha_j}$) at inductive step (i) does not depend on the choice of k (resp. k') such that $m_{i,k} = 3$ (resp. $m_{j,k'} = 3$). So we are free to chose $k = j$ (resp. $k' = i$), and we obtain that both sides of relation (4) are equal to $\mathfrak{d}\left(\mathfrak{f}_{i,p\delta-\alpha_i-\alpha_j} + \mathfrak{f}_{j,p\delta-\alpha_i-\alpha_j}\right) + \frac{1}{\mathfrak{d}}\mathfrak{f}_{2p-1}$ since $\mathfrak{a} + \frac{\mathfrak{b}\mathfrak{c}}{\mathfrak{d}} = \mathfrak{d}$.

Assume finally that it is a relation of type (6)–(10). Lemma 38 shows that the definition at inductive step (ii) does not depend on the choice of the relation, hence we are free to use the considered relation at this step and this gives the result. □

Example 42. The LK-family of Paris (see definition 30) is the one corresponding to the family $(\mathfrak{f}_n)_{n\in\mathbb{N}}$ with $\mathfrak{f}_0 = \mathfrak{f}$ and, for $p \geqslant 1$, $\mathfrak{f}_{2p} = -\dfrac{\mathfrak{a}\mathfrak{f}}{\mathfrak{c}}\left(\dfrac{\mathfrak{b}}{\mathfrak{d}}\right)^{\mathrm{dp}(p\delta+\alpha_i)-2}$ and $\mathfrak{f}_{2p-1} = \dfrac{\mathfrak{a}\mathfrak{d}^2\mathfrak{f}}{\mathfrak{c}}\left(\dfrac{\mathfrak{b}}{\mathfrak{d}}\right)^{\mathrm{dp}(p\delta-\alpha_i)-3}$.

Example 43. Let us assume that $\Gamma = \tilde{A}_n$. Then each $\alpha \in \Phi^+$ has a unique decomposition as $\alpha = p\delta + \sum_{k=j}^{j+\ell} \alpha_{\overline{k}}$, with $p \in \mathbb{N}$, $j \in [\![0,n]\!]$, $\ell \in [\![0,n-1]\!]$, and $\overline{k}$ the rest of k modulo $n+1$. We then call *domain* (resp. *interior*, resp. *boundary*) of α the set $\overline{\alpha} = \{\overline{k} \mid j \leqslant k \leqslant j+\ell\}$ (resp. $\alpha^\circ = \{\overline{k} \mid j+1 \leqslant k \leqslant j+\ell-1\}$, resp. $\partial\alpha = \overline{\alpha} \setminus \alpha^\circ = \{\overline{j}, \overline{j+\ell}\}$). If $(f_i)_{i\in I}$ is an LK-family, then one can check, by induction on $\mathrm{dp}(\alpha)$, that the element $\mathfrak{f}_{i,\alpha} = f_i(e_\alpha)$ is equal to :

- $\mathfrak{f}_{2p}$ if $\alpha = p\delta + \alpha_i$ (*i.e.* $i \in \partial\alpha$ and $\ell = 0$),
- $-\dfrac{\mathfrak{a}}{\mathfrak{c}}\left(\dfrac{\mathfrak{b}}{\mathfrak{d}}\right)^{\ell-1}\displaystyle\sum_{q=0}^{p}\left(\dfrac{\mathfrak{b}^{n-1}}{\mathfrak{c}\mathfrak{d}^{n-2}}\right)^{p-q}\mathfrak{f}_{2q}$ if $i \in \partial\alpha$ and $\ell \geqslant 1$,

- $\Big(\dfrac{\mathfrak{a}}{\mathfrak{c}}\Big)^2\Big(\dfrac{\mathfrak{b}}{\mathfrak{d}}\Big)^{\ell-2}\sum_{q=0}^{p}(p-q+1)\Big(\dfrac{\mathfrak{b}^{n-1}}{\mathfrak{c}\mathfrak{d}^{n-2}}\Big)^{p-q}\mathfrak{f}_{2q}$ if $i\in\alpha^\circ$,
- $\dfrac{\mathfrak{a}^2}{\mathfrak{c}}\Big(\dfrac{\mathfrak{b}}{\mathfrak{d}}\Big)^{\ell}\sum_{q=0}^{p-1}(p-q)\Big(\dfrac{\mathfrak{b}^{n-1}}{\mathfrak{c}\mathfrak{d}^{n-2}}\Big)^{p-q}\mathfrak{f}_{2q}$ if $i\notin\overline{\alpha}$ and $\ell\leqslant n-2$,
- $\dfrac{\mathfrak{a}^2}{\mathfrak{c}}\Big(\dfrac{\mathfrak{b}}{\mathfrak{d}}+\dfrac{\mathfrak{d}}{\mathfrak{c}}\Big)\Big(\dfrac{\mathfrak{b}}{\mathfrak{d}}\Big)^{n-2}\sum_{q=0}^{p-1}(p-q)\Big(\dfrac{\mathfrak{b}^{n-1}}{\mathfrak{c}\mathfrak{d}^{n-2}}\Big)^{p-q}\mathfrak{f}_{2q}+\dfrac{1}{\mathfrak{c}\mathfrak{d}}\mathfrak{f}_{2p-1}$ if $\alpha=p\delta-\alpha_i$ (*i.e.* $i\notin\overline{\alpha}$ and $\ell=n-1$).

3.5. **Comments.**

Assume that we are in the situation of subsection 2.3, that is, in particular, with $\mathfrak{R}=\mathfrak{R}_0[x]$ for some totally ordered commutative ring $\mathfrak{R}_0$.

Then by choosing for $\mathfrak{f}$ an element of $x\mathfrak{R}$ for the basis step of the inductive construction of LK-families in subsections 3.2 and 3.3, it is clear that the obtained LK-family $(f_i)_{i\in i}$ is such that $\mathrm{Im}(f_i)\subseteq x\mathfrak{R}$ for all $i\in I$. We get the same result in the affine case by choosing for $(\mathfrak{f}_n)_{n\in\mathbb{N}}$ a family of elements of $x\mathfrak{R}$ in the inductive construction of subsection 3.4.

If moreover $\mathfrak{f}$ (resp. $\mathfrak{f}_0$) is chosen to be non-zero, hence is a unit of some overring $\mathfrak{R}'$ of $\mathfrak{R}$, then the obtained LK-family will be suitable to apply the faithfulness criterion (theorem 14) to the associated LK-representation ψ of B^+ in order to show that it is faithful.

4. Twisted LK-representations

In [12], Digne defines "twisted" LK-representations for an Artin-Tits group of non-small crystallographic and spherical type (*i.e.* of type B_n, F_4 or G_2), using the fact that this group is the subgroup of fixed elements of an Artin-Tits group of small and spherical type (A_{2n-1}, E_6 or D_4 respectively) under a graph automorphism, and shows that those representations are faithful.

The aim of this section is to generalize this construction and the faithfulness result to any Artin-Tits monoid that appears as the submonoid of fixed elements of an Artin-Tits monoid of small type under a group of graph automorphisms. Note that our proof of faithfulness (*cf.* subsection 4.2) is different from the one of [12] as it is general and avoid any case-by-case analysis.

Let $\Gamma=(m_{i,j})_{i,j\in I}$ be a Coxeter matrix of small type and let G be a subgroup of $\mathrm{Aut}(\Gamma)$. We fix $(\mathfrak{b},\mathfrak{c},\mathfrak{d})\in(\mathfrak{R}^\times)^3$ and set $\mathfrak{a}=\mathfrak{d}-\frac{\mathfrak{b}\mathfrak{c}}{\mathfrak{d}}$. Let us consider an LK-family $(f_i)_{i\in I}\in\mathscr{F}$. Then we get an LK-representation $\psi:B^+\to\mathscr{L}(V)$, $\boldsymbol{s}_i\mapsto\psi_i=\varphi_i+f_i\boxtimes e_{\alpha_i}$, which induces an LK-representation of B (*i.e.* which has invertible images) whenever $(f_i)_{i\in I}\in\mathscr{F}_{\mathrm{gr}}$, *i.e.* $f_i(e_{\alpha_i})\in\mathfrak{R}^\times$ for every $i\in I$.

4.1. **Definition.**

Recall that the group G naturally acts on B^+ and on Φ^+ (see section 1.4). The action of G on Φ^+ induces an action of G on V by permutation of the basis $(e_\alpha)_{\alpha\in\Phi^+}$. We denote by $(B^+)^G$ (resp. V^G) the submonoid (resp. submodule) of fixed points of B^+ (resp. of V) under the action of G. Recall that $(B^+)^G=B^+_{\Gamma'}$ for a certain Coxeter graph Γ'.

We denote by $\varphi:B^+\to\mathscr{L}(V)$, $b\mapsto\varphi_b$, the LK-representation of B^+ associated with the trivial LK-family (*i.e.* with $\psi_i=\varphi_i$ for all $i\in I$).

Lemma 44. *For all* $(b, v, g) \in B^+ \times V \times \mathrm{Aut}(\Gamma)$*, we get* $g(\varphi_b(v)) = \varphi_{g(b)}(g(v))$*. In particular, for all* $b \in (B^+)^G$*,* φ_b *stabilizes* V^G *and hence* φ *induces a linear representation* $\varphi^G : (B^+)^G \to \mathscr{L}(V^G)$*,* $b \mapsto \varphi^G_b = \varphi_b|_{V^G}$*.*

Proof. The action of $\mathrm{Aut}(\Gamma)$ on $E = \oplus_{i\in I}\mathbb{R}\alpha_i$ respects the bilinear form $(\,.\,|\,.\,)_\Gamma$, and this clearly implies that $g(\varphi_i(e_\alpha)) = \varphi_{g(i)}(e_{g(\alpha)})$ in view of the formulas of definition 7. The result follows by linearity and induction on $\ell(b)$. $\square$

Lemma 45. *Assume that* $\mathfrak{f}_{i,\alpha} = \mathfrak{f}_{g(i),g(\alpha)}$ *for every* $(i, \alpha, g) \in I \times \Phi^+ \times G$*. Then for every* i*,* $j \in I$*,* $v \in V$ *and* $g \in G$*, we get* $f_{g(i)}(g(v)) = f_i(v)$ *and* $f_{g(i)}\varphi_{g(j)}(g(v)) = f_i\varphi_j(v)$*. In particular, the linear forms* f_i *and* $f_{g(i)}$ *(resp.* $f_i\varphi_j$ *and* $f_{g(i)}\varphi_{g(j)}$*) coincide on* V^G*.*

Proof. The assumption means that $f_{g(i)}(g(e_\alpha)) = f_i(e_\alpha)$ for every $(i, \alpha, g) \in I \times \Phi^+ \times G$, whence the first point by linearity. The second point follows from the first one and the previous lemma. $\square$

Proposition 46. *Assume that* $\mathfrak{f}_{i,\alpha} = \mathfrak{f}_{g(i),g(\alpha)}$ *for every* $(i, \alpha, g) \in I \times \Phi^+ \times G$*. Then for every* $(b, v, g) \in B^+ \times V \times \mathrm{Aut}(\Gamma)$*, we get* $g(\psi_b(v)) = \psi_{g(b)}(g(v))$*. In particular, for every* $b \in (B^+)^G$*,* ψ_b *stabilizes* V^G *and hence* ψ *induces a linear representation*

$$\psi^G : (B^+)^G \to \mathscr{L}(V^G), \quad b \mapsto \psi^G_b = \psi_b|_{V^G}.$$

Moreover if the images of ψ *are invertible, then so are the images of* ψ^G*.*

Proof. By definition 7, we get $\begin{cases} g(\psi_i(e_\alpha)) = g(\varphi_i(e_\alpha)) + \mathfrak{f}_{i,\alpha} e_{\alpha_{g(i)}}, \text{ and} \\ \psi_{g(i)}(e_{g(\alpha)}) = \varphi_{g(i)}(e_{g(\alpha)}) + \mathfrak{f}_{g(i),g(\alpha)} e_{\alpha_{g(i)}}. \end{cases}$

Whence $g(\psi_i(e_\alpha)) = \psi_{g(i)}(e_{g(\alpha)})$ by assumption and lemma 44, and the first point by linearity and induction on $\ell(b)$. Moreover if the images of ψ are invertible, that is, if $\mathfrak{f}_{i,\alpha_i} = f_i(e_{\alpha_i}) \in \mathfrak{R}^\times$ for every $i \in I$, then the formulas of lemma 9 show that we also get $g(\psi_i^{-1}(e_\alpha)) = \psi^{-1}_{g(i)}(e_{g(\alpha)})$, and hence, similarly to what as just been done, ψ_b^{-1} stabilizes V^G for every $b \in (B^+)^G$. This gives the result. $\square$

Definition 47. Under the assumption of the previous proposition, we call *twisted LK-representation* the linear representation $\psi^G : (B^+)^G \to \mathscr{L}(V^G)$ of the Artin-Tits monoid $(B^+)^G = B^+_{\Gamma'}$, and, when appropriate, the induced linear representation $\psi^G_{\mathrm{gr}} : B_{\Gamma'} \to \mathrm{GL}(V^G)$ of the Artin-Tits group $B_{\Gamma'}$.

The assumption $\mathfrak{f}_{i,\alpha} = \mathfrak{f}_{g(i),g(\alpha)}$ for every $(i, \alpha, g) \in I \times \Phi^+ \times G$ is not always satisfied : for example if i and $g(i)$ are not in the same connected component of Γ, then $\mathfrak{f}_{i,\alpha_i}$ and $\mathfrak{f}_{g(i),\alpha_{g(i)}}$ can be chosen to be distinct (see remark 25 above). I do not know if this assumption is always satisfied when Γ is connected, but we have the following partial result :

Proposition 48. *Let* $(f_i)_{i\in I}$ *be an LK-family, and assume that we are in one of the following cases :*

- (i) Γ *is spherical and irreducible (i.e. of type ADE), or*
- (ii) Γ *is affine (i.e. of type* $\tilde{A}\tilde{D}\tilde{E}$*), or*
- (iii) $(f_i)_{i\in I}$ *is the family constructed in definition 30 and does not depend on the choice of the* j_α*'s (for example if* Γ *has no triangle).*

Then $\mathfrak{f}_{i,\alpha} = \mathfrak{f}_{g(i),g(\alpha)}$ *for every* $(i, \alpha, g) \in I \times \Phi^+ \times \mathrm{Aut}(\Gamma)$*.*

Proof. The result for the three situations (note that the first one is a consequence of the third one) are easy to see by induction on $\mathrm{dp}(\alpha)$, using the inductive construction of $(f_i)_{i\in I}$, and the independence results at the inductive steps, of subsection 3.2, 3.4, or 3.3 respectively, and the fact that the action of $\mathrm{Aut}(\Gamma)$ on $E = \oplus_{i\in I}\mathbb{R}\alpha_i$ respects the bilinear form $(\,.\,|\,.\,)_\Gamma$. $\square$

We denote by Φ^+/G the set of orbits of Φ^+ under G and, for every $\Theta \in \Phi^+/G$, we set $e_\Theta = \sum_{\alpha\in\Theta} e_\alpha$. The family $(e_\Theta)_{\Theta\in\Phi^+/G}$ is a basis of V^G.

4.2. **Twisted faithfulness criterion.**

The aim of this subsection is to prove that the faithfulness criterion of subsection 2.2 also works for a twisted LK-representation ψ^G.

Lemma 49. *Let $\rho : (B^+)^G \to M$ be a monoid homomorphism where M is left cancellative. If ρ satisfies $\rho(b) = \rho(b') \Rightarrow I(b) = I(b')$ for all b, $b' \in (B^+)^G$, then it is injective.*

Proof. Let b, $b' \in (B^+)^G$ be such that $\rho(b) = \rho(b')$. We prove by induction on $\ell(b)$ that $b = b'$. If $\ell(b) = 0$, *i.e.* if $b = 1$, then $I(b) = I(b') = \emptyset$, hence $b' = 1$ and we are done.

If $\ell(b) > 0$, fix $i \in I(b) = I(b')$. Since the action of G on B^+ respects the divisibility and since b is fixed by G, the orbit J of i under G is included in $I(b) = I(b')$, but then J is spherical and there exist b_1, $b_1' \in B^+$ such that $b = \Delta_J b_1$ and $b' = \Delta_J b_1'$. Since J is an orbit of I under G, the element Δ_J is fixed by G and hence so are b_1 and b_1', so we get $\rho(\Delta_J)\rho(b_1) = \rho(\Delta_J)\rho(b_1')$ in M, whence $\rho(b_1) = \rho(b_1')$ by cancellation, therefore $b_1 = b_1'$ by induction and finally $b = b'$. $\square$

Let us assume that the condition of proposition 46 is satisfied, so that the twisted LK-representation $\psi^G : (B^+)^G \to \mathscr{L}(V^G)$ is defined. Then :

Theorem 50. *Assume that the following two conditions are satisfied :*

(i) $\mathrm{Im}(\psi^G)$ *is a left cancellative submonoid of* $\mathscr{L}(V^G)$,
(ii) *there exists a totally ordered commutative ring $\mathfrak{R}_0$ and a ring homomorphism $\mathfrak{R} \to \mathfrak{R}_0$, $\mathfrak{x} \mapsto \overline{\mathfrak{x}}$, such that $\overline{\mathfrak{a}}$, $\overline{\mathfrak{b}}$, $\overline{\mathfrak{c}}$, $\overline{\mathfrak{d}}$ are positive and $\overline{\mathfrak{f}_{i,\alpha}} = 0$ for every $(i,\alpha) \in I \times \Phi^+$.*

Then the twisted LK-representation ψ^G is faithful.

Proof. Note first that, with notations 16, if $R \in \mathrm{Bin}(\Omega)$ and if $(\Psi_\lambda)_{\lambda\in\Lambda}$ is a family of subsets of Ω, we get $R(\bigcup_{\lambda\in\Lambda}\Psi_\lambda) = \bigcup_{\lambda\in\Lambda} R(\Psi_\lambda)$.

In order to prove the theorem, it suffices to show that ψ^G satisfies the assumption of lemma 49. So let b, $b' \in (B^+)^G$ be such that $\psi^G_b = \psi^G_{b'}$ and let us show that $I(b) = I(b')$. Since ψ_b and $\psi_{b'}$ coincide on V^G, we get in particular $\psi_b(e_\Theta) = \psi_{b'}(e_\Theta)$ for every $\Theta \in \Phi^+/G$.

With the notations of definition 19, let us consider the set $R_b(\Theta)$. Since the coefficients of the matrix of $\overline{\psi_b}$ in the basis $(e_\alpha)_{\alpha\in\Phi^+}$ of V_0 are non-negative, the set $R_b(\Theta)$ is precisely the set of those indices $\beta \in \Phi^+$ for which the coefficient of e_β in the decomposition of $\overline{\psi_b}(e_\Theta)$ in the basis $(e_\alpha)_{\alpha\in\Phi^+}$ is positive.

The same occurs for b', and hence $\psi_b(e_\Theta) = \psi_{b'}(e_\Theta)$ implies $R_b(\Theta) = R_{b'}(\Theta)$. Since we have $\Phi^+ = \bigcup_{\Theta\in\Phi^+/G}\Theta$, we thus get $R_b(\Phi^+) = \bigcup_{\Theta\in\Phi^+/G} R_b(\Theta) = \bigcup_{\Theta\in\Phi^+/G} R_{b'}(\Theta) = R_{b'}(\Phi^+)$, and hence $I(b) = I(b')$ by lemmas 17 and 20. $\square$

Remark 51. Assume that we are in the typical situation of subsections 2.3 and 3.5, so that condition (ii) of theorem 50 is satisfied. Then proposition 46 shows that $\mathrm{Im}(\psi^G)$ is included in $\mathscr{L}(V^G) \cap \mathrm{GL}((V')^G)$ and hence condition (i) of theorem 50 is also satisfied. Hence that twisted LK-representation ψ^G is faithful and so is the induced twisted LK-representation $\psi^G_{\mathrm{gr}} : B_{\Gamma'} \to \mathrm{GL}((V')^G)$ when Γ' is spherical.

4.3. **Formulas when $|G| = 2$.**

Recall that $(B^+)^G = B^+_{\Gamma'}$ is generated by the elements Δ_J, for J running through the spherical orbits of I under G (see section 1.4). In this subsection, we assume that $\mathfrak{f}_{i,\alpha} = \mathfrak{f}_{g(i),g(\alpha)}$ for every $(i, \alpha, g) \in I \times \Phi^+ \times G$, and we compute the maps $\psi^G_{\Delta_J}$ for those orbits J, at least when $|G| = 2$.

Notation 52. Let J be an orbit of I under G. Then in view of lemma 45 :

(i) the linear forms f_j, for $j \in J$, coincide on V^G, and we set
$$f_J = f_j|_{V^G} \in (V^G)^\star, \quad \text{for} \quad j \in J,$$
(ii) if $J = \{i, j\}$, then the forms $f_i\varphi_j$ and $f_j\varphi_i$ coincide on V^G, and we set
$$f'_J = f_i\varphi_j|_{V^G} = f_j\varphi_i|_{V^G} \in (V^G)^\star.$$

Note that if J is an orbit of I under G, then $\Theta_J := \{\alpha_i \mid i \in J\}$ is an orbit of Φ^+ under G ; moreover if $J = \{i, j\}$ with $m_{i,j} = 3$, then $\Theta'_J = \{\alpha_i + \alpha_j\}$ is also an orbit of Φ^+ under G.

Proposition 53. *Let J be an orbit of I under G. Then*

(i) *if $J = \{i\}$, $\psi^G_{\Delta_J} = \varphi^G_{\Delta_J} + f_J \boxtimes e_{\Theta_J}$,*
(ii) *if $J = \{i, j\}$ with $m_{i,j} = 2$, $\psi^G_{\Delta_J} = \varphi^G_{\Delta_J} + \mathfrak{d} f_J \boxtimes e_{\Theta_J}$,*
(iii) *if $J = \{i, j\}$ with $m_{i,j} = 3$, $\psi^G_{\Delta_J} = \varphi^G_{\Delta_J} + (\mathfrak{bc} f_J + \mathfrak{a} f'_J) \boxtimes e_{\Theta_J} + \mathfrak{c} f'_J \boxtimes e_{\Theta'_J}$.*

Proof. If $J = \{i\}$, then $\Delta_J = \boldsymbol{s}_i$ and (i) is clear. If $J = \{i, j\}$ with $m_{i,j} = 2$, then $\Delta_J = \boldsymbol{s}_i\boldsymbol{s}_j$ and we get $\psi_i\psi_j = \varphi_i\varphi_j + f_i\varphi_j \boxtimes e_{\alpha_i} + \mathfrak{d} f_j \boxtimes e_{\alpha_j}$ (see the proof of lemma 11), whence (ii) since $f_i\varphi_j = \mathfrak{d} f_i$. Finally if $J = \{i, j\}$ with $m_{i,j} = 3$, then $\Delta_J = \boldsymbol{s}_i\boldsymbol{s}_j\boldsymbol{s}_i$ and we get, following the computations of the proof of lemma 11, $\psi_i\psi_j\psi_i = \varphi_i\varphi_j\varphi_i + f_i\varphi_j\varphi_i \boxtimes e_{\alpha_i} + (\mathfrak{bc} f_i + \mathfrak{a} f_j\varphi_i) \boxtimes e_{\alpha_j} + \mathfrak{c} f_j\varphi_i \boxtimes e_{\alpha_i+\alpha_j}$ (using the fact that $f_i\varphi_j(e_{\alpha_i}) = f_j\varphi_i(e_{\alpha_i}) = 0$), whence (iii) since $f_i\varphi_j\varphi_i|_{V^G} = f_j\varphi_i^2|_{V^G}$ and since we have seen, again in the proof of lemma 11, that $f_j\varphi_i^2 = \mathfrak{bc} f_j + \mathfrak{a} f_j\varphi_i$. ☐

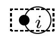
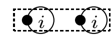
Hence when $|G| = 2$, the previous proposition gives a complete description of the possible maps $\psi^G_{\Delta_J}$ when J runs through the (spherical) orbits of I under G.

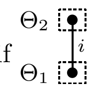
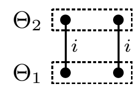
We detail below the matrix of $\varphi^G_{\Delta_J}$ in the basis $(e_\Theta)_{\Theta \in \Phi^+/G}$ and the values on this basis of the linear forms involved in the expression of $\psi^G_{\Delta_J}$. Note that since the maps φ_i, $i \in J$, stabilize the submodule of V generated by the elements e_β for β running through a given J-mesh M, the map φ^G_J stabilizes the submodule of V^G generated by the elements e_Θ for Θ running through the orbits included in $G(M)$. The matrix of φ^G_J in the basis $(e_\Theta)_{\Theta \in \Phi^+/G}$ of V^G is then block diagonal, for the corresponding block decomposition.

4.3.1. *Case $J = \{i\}$.*

- $f_J(e_\Theta) = f_i(e_\Theta) = \begin{cases} \mathfrak{f}_{i,\alpha} & \text{if } \Theta = \{\alpha\} \\ \mathfrak{f}_{i,\alpha} + \mathfrak{f}_{i,\alpha'} = 2\mathfrak{f}_{i,\alpha} & \text{if } \Theta = \{\alpha, \alpha'\}, \alpha \neq \alpha' \end{cases},$
- the blocks of $\varphi^G_{\Delta_J} = \varphi_i|_{V^G}$ are the following ones :

$\overset{e_\Theta}{(0)}$ if $\Theta = \Theta_J$

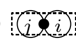
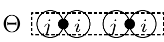
$\overset{e_\Theta}{(\mathfrak{d})}$ if Θ  or Θ 

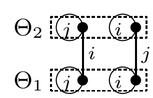
$\begin{array}{c} e_{\Theta_1} \quad e_{\Theta_2} \\ \begin{pmatrix} \mathfrak{a} & \mathfrak{b} \\ \mathfrak{c} & 0 \end{pmatrix} \end{array}$ if  or 

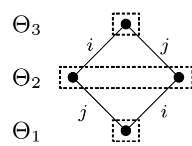
4.3.2. *Case $J = \{i, j\}$ with $m_{i,j} = 2$.*

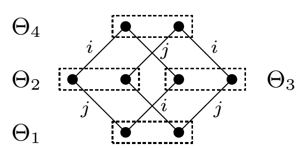
- $f_J(e_\Theta) = \mathfrak{d}\mathfrak{f}_i(e_\Theta) = \begin{cases} \mathfrak{d}\mathfrak{f}_{i,\alpha} & \text{if } \Theta = \{\alpha\} \\ \mathfrak{d}(\mathfrak{f}_{i,\alpha} + \mathfrak{f}_{i,\alpha'}) = \mathfrak{d}(\mathfrak{f}_{i,\alpha} + \mathfrak{f}_{j,\alpha}) & \text{if } \Theta = \{\alpha, \alpha'\}, \alpha \neq \alpha' \end{cases}$,
- the blocks of $\varphi^G_{\Delta_J} = (\varphi_i \varphi_j)|_{V^G}$ are the following ones :

$\overset{e_\Theta}{(0)}$ if $\Theta = \Theta_J$

$\overset{e_\Theta}{(\mathfrak{d}^2)}$ if Θ  or Θ 

$\begin{array}{c} e_{\Theta_1} \quad e_{\Theta_2} \\ \begin{pmatrix} \mathfrak{a}\mathfrak{d} & \mathfrak{b}\mathfrak{d} \\ \mathfrak{c}\mathfrak{d} & 0 \end{pmatrix} \end{array}$ if 

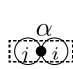
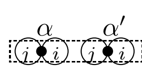
$\begin{array}{c} e_{\Theta_1} \quad e_{\Theta_2} \quad e_{\Theta_3} \\ \begin{pmatrix} \mathfrak{a}^2 & 2\mathfrak{a}\mathfrak{b} & \mathfrak{b}^2 \\ \mathfrak{a}\mathfrak{c} & \mathfrak{b}\mathfrak{c} & 0 \\ \mathfrak{c}^2 & 0 & 0 \end{pmatrix} \end{array}$ if 

$\begin{array}{c} e_{\Theta_1} \quad e_{\Theta_2} \quad e_{\Theta_3} \quad e_{\Theta_4} \\ \begin{pmatrix} \mathfrak{a}^2 & \mathfrak{a}\mathfrak{b} & \mathfrak{a}\mathfrak{b} & \mathfrak{b}^2 \\ \mathfrak{a}\mathfrak{c} & 0 & \mathfrak{b}\mathfrak{c} & 0 \\ \mathfrak{a}\mathfrak{c} & \mathfrak{b}\mathfrak{c} & 0 & 0 \\ \mathfrak{c}^2 & 0 & 0 & 0 \end{pmatrix} \end{array}$ if 

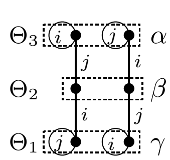
4.3.3. *Case $J = \{i, j\}$ with $m_{i,j} = 3$.*

We detail below the values of the linear forms $\mathfrak{b}\mathfrak{c}f_J + \mathfrak{a}f'_J$ and $\mathfrak{c}f'_J$, and the blocks of $\varphi^G_{\Delta_J} = (\varphi_i \varphi_j \varphi_i)|_{V^G}$ for the different possible configurations of orbits.
- Orbits $\Theta_J = \{\alpha_i, \alpha_j\}$ and $\Theta'_J = \{\alpha_i + \alpha_j\}$.

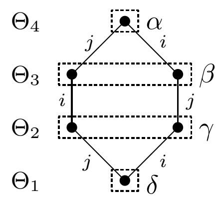
Orbits	Values of $\mathfrak{b}\mathfrak{c}f_J + \mathfrak{a}f'_J$	Values of $\mathfrak{c}f'_J$	Block in $\varphi^G_{\Delta_J}$
Θ_J Θ'_J	$\mathfrak{b}\mathfrak{c}\mathfrak{f}_{i,\alpha_i}$ 0	0 $\mathfrak{b}\mathfrak{c}\mathfrak{f}_{i,\alpha_i}$	$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

- Configuration Θ  or Θ 

Value of $\mathfrak{b}\mathfrak{c}f_J + \mathfrak{a}f'_J$	Value of $\mathfrak{c}f'_J$	Block in $\varphi^G_{\Delta_J}$
$\lvert\Theta\rvert\mathfrak{d}^2\mathfrak{f}_{i,\alpha}$	$\lvert\Theta\rvert\mathfrak{c}\mathfrak{d}\mathfrak{f}_{i,\alpha}$	$\overset{e_\Theta}{(\mathfrak{d}^3)}$

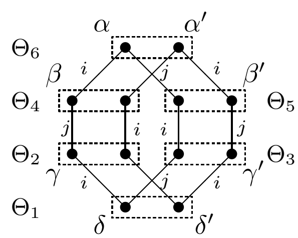
- Configuration 

Orbits	Values of $\mathfrak{b}\mathfrak{c}f_J+\mathfrak{a}f'_J$	Values of $\mathfrak{c}f'_J$	Block in $\varphi^G_{\Delta_J}$
Θ_1	$\mathfrak{b}\mathfrak{c}\mathfrak{f}_{i,\gamma}+\mathfrak{d}(\mathfrak{d}+\mathfrak{a})\mathfrak{f}_{j,\gamma}$	$2\mathfrak{c}\mathfrak{d}\mathfrak{f}_{j,\gamma}$	$\begin{array}{c} \begin{array}{ccc} e_{\Theta_1} & e_{\Theta_2} & e_{\Theta_3} \end{array} \\ \begin{pmatrix} \mathfrak{a}\mathfrak{d}^2 & \mathfrak{a}\mathfrak{b}\mathfrak{d} & \mathfrak{b}^2\mathfrak{d} \\ \mathfrak{a}\mathfrak{c}\mathfrak{d} & \mathfrak{b}\mathfrak{c}\mathfrak{d} & 0 \\ \mathfrak{c}^2\mathfrak{d} & 0 & 0 \end{pmatrix} \end{array}$
Θ_2	$\mathfrak{b}\left(\mathfrak{c}\mathfrak{f}_{j,\beta}+\mathfrak{a}\mathfrak{f}_{i,\gamma}+\mathfrak{d}\mathfrak{f}_{j,\gamma}\right)$	$2\mathfrak{b}\mathfrak{c}\mathfrak{f}_{i,\gamma}$	
Θ_3	$\mathfrak{b}\left(\mathfrak{d}\mathfrak{f}_{j,\beta}+\mathfrak{b}\mathfrak{f}_{i,\gamma}\right)$	$2\mathfrak{b}\mathfrak{c}\mathfrak{f}_{j,\beta}$	

• Configuration 

Orbits	Values of $\mathfrak{b}\mathfrak{c}f_J+\mathfrak{a}f'_J$	Values of $\mathfrak{c}f'_J$
Θ_1	$\mathfrak{a}\mathfrak{c}\mathfrak{f}_{j,\gamma}+(\mathfrak{a}^2+\mathfrak{b}\mathfrak{c})\mathfrak{f}_{i,\delta}$	$\mathfrak{c}^2\mathfrak{f}_{j,\gamma}+\mathfrak{a}\mathfrak{c}\mathfrak{f}_{j,\delta}$
Θ_2	$\mathfrak{b}\left(\mathfrak{c}(\mathfrak{f}_{i,\gamma}+\mathfrak{f}_{j,\gamma})+2\mathfrak{a}\mathfrak{f}_{i,\delta}\right)$	$2\mathfrak{b}\mathfrak{c}\mathfrak{f}_{i,\delta}$
Θ_3	$\mathfrak{b}\left(\mathfrak{c}\mathfrak{f}_{j,\beta}+\mathfrak{a}\mathfrak{f}_{i,\gamma}+\mathfrak{b}\mathfrak{f}_{i,\delta}\right)$	$2\mathfrak{b}\mathfrak{c}\mathfrak{f}_{i,\gamma}$
Θ_4	$\mathfrak{b}^2\mathfrak{f}_{i,\gamma}$	$\mathfrak{b}\mathfrak{c}\mathfrak{f}_{j,\beta}$

Block in $\varphi^G_{\Delta_J}$
$\begin{array}{c} \begin{array}{cccc} e_{\Theta_1} & e_{\Theta_2} & e_{\Theta_3} & e_{\Theta_4} \end{array} \\ \begin{pmatrix} \mathfrak{a}(\mathfrak{a}^2+\mathfrak{b}\mathfrak{c}) & 2\mathfrak{a}^2\mathfrak{b} & 2\mathfrak{a}\mathfrak{b}^2 & \mathfrak{b}^3 \\ \mathfrak{a}^2\mathfrak{c} & 2\mathfrak{a}\mathfrak{b}\mathfrak{c} & \mathfrak{b}^2\mathfrak{c} & 0 \\ \mathfrak{a}\mathfrak{c}^2 & \mathfrak{b}\mathfrak{c}^2 & 0 & 0 \\ \mathfrak{c}^3 & 0 & 0 & 0 \end{pmatrix} \end{array}$

• Configuration 

Orbits	Values of $\mathfrak{b}\mathfrak{c}f_J+\mathfrak{a}f'_J$	Values of $\mathfrak{c}f'_J$
Θ_1	$\mathfrak{b}\mathfrak{c}\left(\mathfrak{f}_{i,\delta}+\mathfrak{f}_{j,\delta}\right)+2\mathfrak{a}\left(\mathfrak{a}\mathfrak{f}_{j,\delta}+\mathfrak{c}\mathfrak{f}_{j,\gamma}\right)$	$2\mathfrak{c}(\mathfrak{a}\mathfrak{f}_{j,\delta}+\mathfrak{c}\mathfrak{f}_{j,\gamma})$
Θ_2	$\mathfrak{b}\left(\mathfrak{c}(\mathfrak{f}_{i,\gamma}+\mathfrak{f}_{j,\gamma})+2\mathfrak{a}\mathfrak{f}_{i,\delta'}\right)$	$2\mathfrak{b}\mathfrak{c}\mathfrak{f}_{i,\delta'}$
Θ_3	$\mathfrak{b}\left(\mathfrak{c}(\mathfrak{f}_{i,\gamma'}+\mathfrak{f}_{j,\gamma'})+2\mathfrak{a}\mathfrak{f}_{i,\delta}\right)$	$2\mathfrak{b}\mathfrak{c}\mathfrak{f}_{i,\delta}$
Θ_4	$\mathfrak{b}\left(\mathfrak{c}\mathfrak{f}_{j,\beta}+\mathfrak{a}\mathfrak{f}_{i,\gamma}+\mathfrak{b}\mathfrak{f}_{j,\delta}\right)$	$2\mathfrak{b}\mathfrak{c}\mathfrak{f}_{i,\gamma}$
Θ_5	$\mathfrak{b}\left(\mathfrak{c}\mathfrak{f}_{j,\beta'}+\mathfrak{a}\mathfrak{f}_{i,\gamma'}+\mathfrak{b}\mathfrak{f}_{j,\delta'}\right)$	$2\mathfrak{b}\mathfrak{c}\mathfrak{f}_{i,\gamma'}$
Θ_6	$\mathfrak{b}^2(\mathfrak{f}_{i,\gamma}+\mathfrak{f}_{i,\gamma'})$	$2\mathfrak{b}\mathfrak{c}\mathfrak{f}_{j,\beta}$

Block in $\varphi^G_{\Delta_J}$
$\begin{array}{c} \begin{array}{cccccc} e_{\Theta_1} & e_{\Theta_2} & e_{\Theta_3} & e_{\Theta_4} & e_{\Theta_5} & e_{\Theta_6} \end{array} \\ \begin{pmatrix} \mathfrak{a}(\mathfrak{a}^2+\mathfrak{b}\mathfrak{c}) & \mathfrak{a}^2\mathfrak{b} & \mathfrak{a}^2\mathfrak{b} & \mathfrak{a}\mathfrak{b}^2 & \mathfrak{a}\mathfrak{b}^2 & \mathfrak{b}^3 \\ \mathfrak{a}^2\mathfrak{c} & \mathfrak{a}\mathfrak{b}\mathfrak{c} & \mathfrak{a}\mathfrak{b}\mathfrak{c} & 0 & \mathfrak{b}^2\mathfrak{c} & 0 \\ \mathfrak{a}^2\mathfrak{c} & \mathfrak{a}\mathfrak{b}\mathfrak{c} & \mathfrak{a}\mathfrak{b}\mathfrak{c} & \mathfrak{b}^2\mathfrak{c} & 0 & 0 \\ \mathfrak{a}\mathfrak{c}^2 & 0 & \mathfrak{b}\mathfrak{c}^2 & 0 & 0 & 0 \\ \mathfrak{a}\mathfrak{c}^2 & \mathfrak{b}\mathfrak{c}^2 & 0 & 0 & 0 & 0 \\ \mathfrak{c}^3 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}$

4.4. **Twisted LK-representations of type B.**

Fix $n \in \mathbb{N}_{\geqslant 3}$. Then the Artin-Tits group B of Coxeter type B_n appears as the subgroup of fixed elements of three Artin-Tits groups of type $\Gamma_1 = A_{2n-1}$, $\Gamma_2 = A_{2n}$ and $\Gamma_3 = D_{n+1}$ respectively, under a group of graph automorphisms G_1, G_2 and G_3 respectively, where $G_k = \mathrm{Aut}(\Gamma_k)$ for $1 \leqslant k \leqslant 3$, except for $n = k = 3$ where G_3 is a subgroup of order two of $\mathrm{Aut}(D_4)$ (see [9, 7]). We denote by I_k the vertex set of Γ_k for $1 \leqslant k \leqslant 3$.

Now fix a commutative ring $\mathfrak{R}$ and $(\mathfrak{b}, \mathfrak{c}, \mathfrak{d}) \in (\mathfrak{R}^\times)^3$, and consider three LK-representations $\psi_1 : B_{\Gamma_1} \to \mathrm{GL}(V_1)$, $\psi_2 : B_{\Gamma_2} \to \mathrm{GL}(V_2)$ and $\psi_3 : B_{\Gamma_3} \to \mathrm{GL}(V_3)$. Recall that ψ_k, $1 \leqslant k \leqslant 3$, is determined by the common value $\mathfrak{f}_k \in \mathfrak{R}^\times$ of the $\mathfrak{f}_{i,\alpha_i} = f_i(e_{\alpha_i})$ for $i \in I_k$ (see subsection 3.2). In view of propositions 46 and 48 above, we get three twisted LK-representations $\psi_1^{G_1} : B \to \mathrm{GL}(V_1^{G_1})$, $\psi_2^{G_2} : B \to \mathrm{GL}(V_2^{G_2})$ and $\psi_3^{G_3} : B \to \mathrm{GL}(V_3^{G_3})$ of the Artin-Tits group B. Note that $\psi_1^{G_1}$ is essentially the representation of B considered in [12].

The representation $\psi_2^{G_2}$ is trivially non-equivalent to the two others since it is of degree $|\Phi^+_{\Gamma_2}/G_2| = n(n+1)$ whereas the two others are of degree $|\Phi^+_{\Gamma_1}/G_1| = |\Phi^+_{\Gamma_3}/G_3| = n^2$. The aim of this section is to show that $\psi_1^{G_1}$ and $\psi_3^{G_3}$ are non-equivalent, when $\mathfrak{R} = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ and $(\mathfrak{b}, \mathfrak{c}, \mathfrak{d}) = (y^p, y^q, y^r)$ with p, q, $r \in \mathbb{Z}$ such that $2r \neq p + q$ (as in subsection 2.3), and at least for all $n \geqslant 3$ but two.

Notation 54. Following [2], we label by 1, 2, $\ldots, n$, the vertices of the Coxeter graph B_n, in such a way that the vertex n is the terminal vertex of the edge labeled 4, and we denote by $\Delta_1, \ldots, \Delta_n$ the corresponding standard generators of B.

Note that we will keep the same symbols for the standard generators of B when considering this group as an abstract Artin-Tits group, or as the subgroup of fixed elements $(B_{\Gamma_k})^{G_k}$ of B_{Γ_k} for $1 \leqslant k \leqslant 3$. The meaning of Δ_i, $1 \leqslant i \leqslant n$, in terms of a product of the standard generators of B_{Γ_k}, $1 \leqslant k \leqslant 3$, is given in the following table (where the vertices of Γ_k are labeled as in [2]) :

	$1 \leqslant i < n$	$i = n$
$k=1$	$\boldsymbol{s}_i\boldsymbol{s}_{2n-i}$	$\boldsymbol{s}_n$
$k=2$	$\boldsymbol{s}_i\boldsymbol{s}_{2n+1-i}$	$\boldsymbol{s}_n\boldsymbol{s}_{n+1}\boldsymbol{s}_n$
$k=3$	$\boldsymbol{s}_i$	$\boldsymbol{s}_n\boldsymbol{s}_{n+1}$

Lemma 55. *The determinant of the map $(\psi_k^{G_k})_{\Delta_i}$, for $1 \leqslant i \leqslant n$ and $1 \leqslant k \leqslant 3$, is given in the following table :*

	$1 \leqslant i < n$	$i = n$
$k=1$	$-(\mathfrak{b}\mathfrak{c})^{2n-1}\mathfrak{d}^{2n(n-2)}\mathfrak{f}_1$	$(-1)^{n-1}(\mathfrak{b}\mathfrak{c})^{n-1}\mathfrak{d}^{(n-1)^2}\mathfrak{f}_1$
$k=2$	$(\mathfrak{b}\mathfrak{c})^{2n}\mathfrak{d}^{2(n^2-n-1)}\mathfrak{f}_2$	$(-1)^{n-1}(\mathfrak{b}\mathfrak{c})^{3n-1}\mathfrak{d}^{3n(n-1)}\mathfrak{f}_2^2$
$k=3$	$-(\mathfrak{b}\mathfrak{c})^{2n-3}\mathfrak{d}^{(n-1)^2}\mathfrak{f}_3$	$(-1)^{n-1}(\mathfrak{b}\mathfrak{c})^{3(n-1)}\mathfrak{d}^{2(n-1)(n-2)}\mathfrak{f}_3$

Proof. In view of the formulas of subsection 4.3, the determinant of $(\psi_k^{G_k})_{\Delta_i}$ is of the form $\det(M)\mathfrak{f}_k$ (resp. $\det(M)(\mathfrak{b}\mathfrak{c}\mathfrak{f}_2)^2$) if $(i,k) \neq (n,2)$ (resp. $(i,k) = (n,2)$), where M is a block diagonal matrix, with blocks of determinant $\mathfrak{d}$, $-\mathfrak{b}\mathfrak{c}$, $\mathfrak{d}^2$, $-\mathfrak{b}\mathfrak{c}\mathfrak{d}^2$, $-(\mathfrak{b}\mathfrak{c})^3$ or $(\mathfrak{b}\mathfrak{c})^4$ (resp. $\mathfrak{d}^3$, $-(\mathfrak{b}\mathfrak{c}\mathfrak{d})^3$, $(\mathfrak{b}\mathfrak{c})^6$ or $-(\mathfrak{b}\mathfrak{c})^9$) depending on the configuration of the corresponding orbit in $\Phi^+_{\Gamma_k}$. The result then follows from a direct computation of the number of occurrences of each configuration in $\Phi^+_{\Gamma_k}$ for $1 \leqslant k \leqslant 3$. $\square$

Proposition 56. *Assume that* $\mathfrak{R} = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}]$ *and that* $(\mathfrak{b}, \mathfrak{c}, \mathfrak{d}) = (y^p, y^q, y^r)$, *with* p, q, $r \in \mathbb{Z}$ *such that* $2r \neq p+q$. *Then the twisted LK-representations* $\psi_1^{G_1}$ *and* $\psi_3^{G_3}$ *are non-equivalent, except possibly for two values of* n *when* $r < 0 < p+q$ *or* $p+q < 0 < r$.

Proof. If $\psi_1^{G_1}$ and $\psi_3^{G_3}$ were equivalent, then the determinants of $(\psi_1^{G_1})_{\Delta_i}$ and $(\psi_3^{G_3})_{\Delta_i}$ for $1 \leqslant i < n$ (resp. of $(\psi_1^{G_1})_{\Delta_n}$ and $(\psi_3^{G_3})_{\Delta_n}$) should be equal. This would imply, in view of the previous lemma,

$$\begin{cases} (\mathfrak{bc})^2 \mathfrak{d}^{n^2-2n-1} \mathfrak{f}_1 = \mathfrak{f}_3, \text{ and} \\ \mathfrak{f}_1 = (\mathfrak{bc})^{2(n-1)} \mathfrak{d}^{(n-1)(n-3)} \mathfrak{f}_3 \end{cases}, \text{ whence } (\mathfrak{bc})^{2n} \mathfrak{d}^{2(n^2-3n+1)} = 1,$$

and by choice of $(\mathfrak{b}, \mathfrak{c}, \mathfrak{d})$, this is equivalent to $2n(p+q) + 2(n^2-3n+1)r = 0$.

It is clear that there is at most two values of n satisfying this equality, and that in such a case, r and $p+q$ cannot be zero or of the same sign (note that $n^2 - 3n + 1 = (n-1)(n-2) - 1 \geqslant 1$ since $n \geqslant 3$). □

4.5. **Final remark on Φ^+/G.**

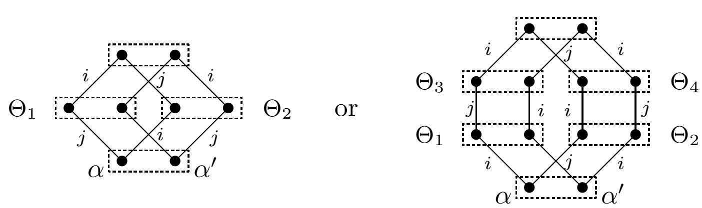
Recall that we denote by Γ' the type of W^G and $(B^+)^G$.

When Γ is spherical, it is possible to index the basis $(e_\Theta)_{\Theta \in \Phi^+/G}$ of V^G with the set of positive roots of a *finite crystallographic* root system (*i.e.* a root system in the sense of [2, Ch. VI]) of Weyl group $W_{\Gamma'}$, via the bijection $\Theta \mapsto \alpha_\Theta$, where $\alpha_\Theta = \dfrac{1}{\mathrm{Card}(\Theta)} \sum_{\alpha \in \Theta} \alpha$ (see [5, Ch. 13] for justifications).

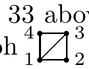
For example if $\Gamma = A_{2n-1}$, A_{2n} or D_{n+1} and $G = \mathrm{Aut}(\Gamma)$ (or a subgroup of order 2 of $\mathrm{Aut}(\Gamma)$ for D_4), we get a finite crystallographic root system of Dynkin type C_n, BC_n or B_n respectively.

This change of index set increases the resemblance between the twisted and non-twisted cases, and has been used by Digne in [12] for his proof of faithfulness of ψ^G, in the particular cases $\Gamma = A_{2n-1}$, E_6 or D_4 and $G = \mathrm{Aut}(\Gamma)$.

But this change of index set is not possible in general. Indeed, the map $\Theta \mapsto \alpha_\Theta$ is not necessarily injective if Γ is not spherical : for example when $|G| = 2$, then for the following configurations of orbits



we get $\alpha_{\Theta_1} = \alpha_{\Theta_2}$ and $\alpha_{\Theta_3} = \alpha_{\Theta_4}$ as soon as $(\alpha_i|\alpha) = (\alpha_j|\alpha)$.

Note that the first of those counterexamples occurs for example in a root system of type $\tilde{A}_{2n-1}$ ($n \geqslant 2$) with G generated by the "half turn", and the second (which does not occur in the affine cases in view of remark 33 above) occurs for example in the root system associated with the Coxeter graph 

with $G = \langle (1\ 3)(2\ 4) \rangle$, $\{i, j\} = \{1, 3\}$ and $\{\alpha, \alpha'\} = \{\alpha_2, \alpha_4\}$.

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